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An axiomatic characterization of the Hirsch-index

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ABSTRACT

The Hirsch-index is a well-known index for measuring and comparing the output of scientific researchers. The main contribution of this article is an axiomatic characterization of the Hirsch-index in terms of three natural axioms. Furthermore, two other scientific impact indices (called the w-index and the maximum-index) are defined and characterized in terms of similar axioms.

1. Introduction

How does one measure productivity, quality, and visibility of a scientific researcher? How does one quantify the cumulative impact and relevance of an individual's scientific research output? In our current academic system, many crucial decisions around faculty recruitment, research time, Ph.D. positions, travel money, award of grants, and promotions depend on our answers to these questions. Natural approaches are based on the publication records and the citation records. Relevant parameters are for instance the number of published papers, the number of citations for each paper, the journals where the papers were published, the impact factors of these journals, etc.

In 2005, Jorge Hirsch proposed the so-called *Hirsch-index* (or *h-index*) to quantify both the scientific productivity and the scientific impact of a scientist. This Hirsch-index is based on the scientist's most cited papers and on the number of citations that they have received in other people's publications: *"A scientist has index h, if h of his or her n papers have at least h citations each, and the other $n - h$ papers have at most h citations each."* Hence a scientist with a Hirsch-index of 25 has published 25 papers that have each attracted at least 25 citations; some of these papers may have attracted considerably more

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than 25 citations, and other publications of this author may have attracted considerably fewer than 25 citations. Since the Hirsch-index reflects both the number of publications and the number of citations per publication, authors with very few high-impact publications and authors with many low-impact publications will score a weak Hirsch-index. Obviously, a scientist's Hirsch-index can never decrease over time, but may well increase as new papers are published and the old papers incrementally attract citations. Hirsch (2005) argues that two individuals with similar Hirsch-index are comparable in terms of their overall scientific impact, even if their total number of papers or their total number of citations is very different. Conversely, comparing two individuals (of the same scientific age) with a similar number of papers or a similar citation count but very different Hirsch-index, the one with the higher Hirsch-index is likely to be the more accomplished scientist.

Hirsch (2005) demonstrates that the Hirsch-index has high predictive value for theoretical physicists: For instance, Nobel prize winners in this area usually have a Hirsch-index between 35 and 39, and over the last 20 years every Nobel prize winner had a Hirsch-index between 22 and 79. Since different research areas have different publishing cultures, the Hirsch-index can not be used to compare researchers from different fields. For instance, the Hirsch-index of a moderately productive scientist in physics typically equals his number of years of service, whereas the Hirsch-index of a biomedical scientist tends to be substantially higher. Over the last few years, the Hirsch-index has become widely used and recognized. Cronin and Meho (2006) and Oppenheim (2007) apply it to rank influential information scientists. Bornmann and Daniel (2005, 2007), Hirsch (2007), and van Raan (2006) study and compare the Hirsch-index against other bibliometric indicators.

In this paper, we provide an axiomatic characterization of the Hirsch-index, in very much the same spirit as Arrow (1950, 1951), May (1952), and Moulin (1988) did for numerous other problems in mathematical decision making. Furthermore, we will define and analyze two other scientific impact indices that we dub the w-index and the maximum-index. We feel that the w-index is not only of theoretical interest, but may be useful for practical purposes.

This article is organized as follows: Section 2 provides basic definitions around scientific impact indices. Section 3 introduces five natural axioms that capture certain desired features of scientific impact indices. Section 4 formulates our axiomatic characterizations of the Hirsch-index, the w-index, and the maximum-index, and Section 5 contains the proofs of these characterizations.

2. Scientific impact indices

A researcher with $n \geq 0$ publications is formally described by a vector $x = (x_1, \ldots, x_n)$ with non-negative integer components $x_1 \geq x_2 \geq \cdots \geq x_n$; the kth component x_k of this vector states the total number of citations to this researcher's kth-most important publication. If $n = 0$, the researcher has no publications and the vector is empty. Let X denote the set of all such vectors. We say that a vector $x = (x_1, \ldots, x_n)$ is *dominated* by a vector $y = (y_1, \ldots, y_m)$, if $n \leq m$ holds and if $x_k \leq y_k$ for $1 \leq k \leq n$; we will write $x \preceq y$ to denote this situation.

Definition 2.1. A *scientific impact index* (or index, for short) is a function f from the set X into the set $\mathbb{N}$ of non-negative integers that satisfies the following two conditions:

- If $x = (0, 0, \ldots, 0)$ or if x is the empty vector, then $f(x) = 0$.
- Monotonicity: If $x \preceq y$, then $f(x) \leq f(y)$.

Both properties seem to be a *conditio sine qua non* for measuring the scientific impact: If one's scientific research fails to generate citations (all-zero vector x), it has no impact. A researcher without publications (empty vector x) has no impact. If the citations to the scientific output of researcher Y dominate the citations to the scientific output of researcher X publication by publication, then Y has more impact than X. The monotonicity condition also implies that the scientific impact of a scientist can never decrease over time.

The following two definitions provide a formal mathematical description of the Hirsch-index, and introduce a (closely related) new scientific impact index that we will call the w-index. An h-index of at least k means that there are k distinct publications that all have at least k citations. And a w-index

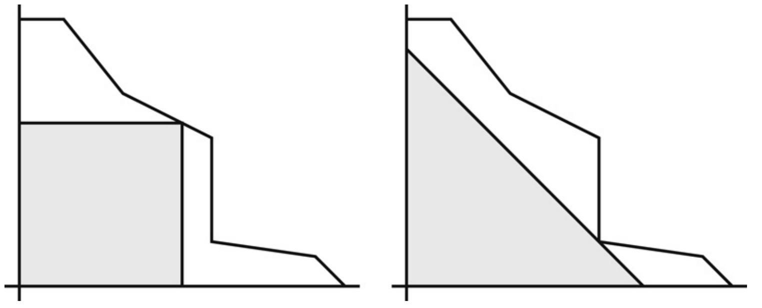

Fig. 1. Publications are on the horizontal axis, and the numbers $x_1 \geq x_2 \geq \cdots \geq x_n$ of citations per publication are on the vertical axis. The area under the curve gives the total number of citations. The side of the square on the left hand side yields the h-index. The legs of the isosceles right-angled triangle on the right hand side yield the w-index.

of at least k means that there are k distinct publications that have at least $1, 2, 3, 4, \ldots, k$ citations, respectively.

Definition 2.2. The h-index (or Hirsch-index) is the scientific impact index $h : X \to \mathbb{N}$ that assigns to vector $x = (x_1, \ldots, x_n)$ the value $h(x) := \max\{k : x_m \geq k \text{ for all } m \leq k\}$.

Definition 2.3. The w-index is the scientific impact index $w : X \to \mathbb{N}$ that assigns to vector $x = (x_1, \ldots, x_n)$ the value $w(x) := \max\{k : x_m \geq k - m + 1 \text{ for all } m \leq k\}$.

Equivalently, the h-index can be defined as $h(x) := \max\{k : x_k \geq k\}$. We prefer the formulation in Definition 2.2 since it clearly exhibits the similarity to the w-index.

Fig. 1 provides two geometric illustrations for the h-index and the w-index. The depicted curve plots the citations of n publications in decreasing order. The h-index (illustrated on the left hand side) corresponds to a largest possible $h \times h$ square below this curve. This square has one corner at the origin, and its diametric corner is the intersection point of the curve with the 45° line through the origin. The w-index (illustrated on the right hand side) corresponds to a largest possible isosceles right-angled triangle below the curve. This isosceles triangle has one corner at the origin, one leg of length w on the horizontal axis, and another leg of length w on the vertical axis. The longest side of the triangle touches the curve. (Sloppily speaking, the h-index maximizes the volume of a scaled copied of an ℓ_∞ unit ball under the curve, while the w-index maximizes the volume of a scaled copied of an ℓ_1 unit ball under the curve.)

Apparently, the h-index and the w-index are closely related, and at most a factor of 2 away from each other – a square of side length h contains an isosceles right-angled triangle with legs of length h. And an isosceles right-angled triangle with legs of length w contains a square of side length $w/2$. This yields the following proposition.

Proposition 2.4. *Every vector $x \in X$ satisfies $h(x) \leq w(x) \leq 2\,h(x)$.*

It is easy to determine one's h-index and w-index manually by using one of the free Internet databases (as for instance *Google Scholar*). The two computation procedures are very similar and kind of dual to each other. The only difference is that one computation traverses the publications in increasing order of citations, whereas the other computation does this in decreasing order.

> The h-index is determined as follows. Work through the list of publications in DECreasing order of citations (as $x_1 \geq x_2 \geq \cdots \geq x_n$), and keep a counter CTR that is initialized at 0. Every time you move to a new x_k, compare it against the current value of the counter. If CTR $< x_k$, then increase CTR , otherwise do nothing. Once you reach your bottom publication x_n, the value of the counter contains your h-index.
>
> The w-index is determined as follows. Work through the list of publications in INCreasing order of citations (as $x_n \leq x_{n-1} \leq \cdots \leq x_1$), and keep a counter CTR that is initialized at 0. Every

time you move to a new x_k, compare it against the current value of the counter. If CTR $< x_k$, then increase CTR , otherwise do nothing. Once you reach your top publication x_1, the value of the counter contains your w-index.

When computing the h-index, the above procedure may actually be terminated early as soon as CTR $\geq x_k$ holds for the first time (since the counter will never be increased after this point). Therefore in practice the w-index will take longer to compute, as senior authors typically have dozens of articles, that is, as n is typically much larger than $h(x)$ and $w(x)$.

To conclude this section, we introduce the so-called *maximum-index* $f_{\max}$ which simply counts the number of citations to one's strongest publication.

Definition 2.5. The maximum-index is the scientific impact index $f_{\max} : X \to \mathbb{N}$ that assigns to vector $x = (x_1, \ldots, x_n)$ the value $f_{\max}(x) := x_1$.

Obviously every vector $x \in X$ satisfies $h(x) \leq f_{\max}(x)$ and $w(x) \leq f_{\max}(x)$. There is no non-trivial way of lower-bounding the h-index and the w-index in terms of the maximum-index.

3. The five axioms

We will formulate five fairly natural axioms that capture certain desired elementary properties of a scientific impact index $f : X \to \mathbb{N}$. The first two axioms concern the addition of a single publication to a publication list. If the publication is only average with respect to the current index, it should not raise the index. But if the publication is above the current average, then this should also be reflected in a higher index.

A1. If the $(n+1)$-dimensional vector y results from the n-dimensional vector x by adding a new article with $f(x)$ citations, then $f(y) \leq f(x)$.

A2. If the $(n+1)$-dimensional vector y results from the n-dimensional vector x by adding a new article with $f(x) + 1$ citations, then $f(y) > f(x)$.

The next two axioms concern the addition of new citations to old publications. Minor changes in the citation record should not lead to major changes in the index. One such minor change is that one single publication receives more citations.

B. If the n-dimensional vector y results from the n-dimensional vector x by increasing the number of citations of a single article, then $f(y) \leq f(x) + 1$.

Another minor change occurs, if every publication in the publication record receives at most one additional citation.

C. If the n-dimensional vector y results from the n-dimensional vector x by increasing the number of citations of every article by at most one, then $f(y) \leq f(x) + 1$.

Our final axiom concerns the case where both, the number of publication (as in A1 and A2) and the number of citations (as in B and C) go up. Adding a strong new publication and consistently improving the citations to one's old publications should also raise the index.

D. If the $(n+1)$-dimensional vector y results from the n-dimensional vector x by first adding an article with $f(x)$ citations and afterwards increasing the number of citations of every article by at least one, then $f(y) > f(x)$.

We stress that the three axioms A1, A2, and D should be interpreted within the context of the monotonicity condition in Definition 2.1. These three axioms discuss the addition of a new article with $f(x) + 1$ or $f(x)$ citations, and monotonicity generalizes this to the addition of an article with *at most* $f(x)$ citations (for axiom A1), *at least* $f(x) + 1$ citations (for axiom A2), and *at least* $f(x)$ citations (for axiom D).

It is easy to see that the five axioms are not independent of each other. For instance axiom A2 implies axiom D.

Proposition 3.1. *If a scientific impact index satisfies axiom* A2*, then it also satisfies* D.

A more interesting dependency is stated in the following theorem; its proof will be delayed to Section 5.

Theorem 3.2. *If a scientific impact index satisfies the axioms* A1 *and* A2*, then it must violate axiom* B.

As an immediate consequence of Theorem 3.2, we derive that the five axioms are contradictory.

Proposition 3.3. *No scientific impact index can simultaneously satisfy all five axioms* A1, A2, B, C*, and* D.

Finally, let us discuss the contents of these axioms. Axioms A1 and A2 are compelling, and precisely express one's intuition as to how a scientific impact index should behave. Also axioms C and D do not seem to need any further justification. They express the desired sensitivity of an index against minor changes in the citation record in a perfectly natural way.

Axiom B on the other hand is worthy of more comments. Consider, for instance, the researcher John Forbes Nash who only published three vastly influential pieces in game theory before leaving the field (the articles on equilibrium points in n-person games, on the bargaining problem, and on two-person cooperative games). In 1994, John Nash was awarded the Nobel prize in Economics for his contributions to game theory. The huge impact of Nash's work in Economics is beyond discussion. Hence it is not at all obvious that a single very successful publication should never allow one's index to take off, as axiom B imposes. We have two comments on this. First, researchers like John Nash are exceptional cases and outliers. There is no universally perfect index (as Proposition 3.3 indicates), and for every index there will be examples on which it performs poorly. Secondly, the results of this article (Theorem 4.3) demonstrate that without axiom B, we are promptly led to the unappealing maximum-index.

4. Axiomatic characterizations

We will analyze the h-index, the w-index, and the maximum-index in terms of the five axioms A1, A2, B, C, and D. The proofs of all theorems can be found in Section 5.

Let us begin our discussion with the Hirsch-index: The h-index satisfies the four axioms A1, B, C, and D. Unfortunately, it violates the very natural axiom A2. For instance, the two vectors $x = (5, 4, 3, 2, 1)$ and $y = (6, 5, 4, 3, 2, 1)$ with $h(x) = h(y) = 3$ collide with axiom A2. The main result of this article is the following axiomatic characterization of the h-index in terms of three axioms.

Theorem 4.1. *A scientific impact index $f : X \to \mathbb{N}$ satisfies the three axioms* A1, B*, and* D*, if and only if it is the h-index.*

We remark that in the statement of Theorem 4.1, axiom D could actually be replaced by the following weaker axiom D′: *"If for some $k \geq 1$ the vector x consists of exactly k articles with exactly k citations, then $f(x) = k$."* In the proof of Theorem 4.1, we will actually only use the D′-part of axiom D. On the other hand axiom D′ seems somewhat artificial, and we see little motivation why it should be imposed explicitly.

Now let us turn to the w-index. The w-index satisfies the four axioms A2, B, C, D, and hence performs better than the h-index with respect to axiom A2. With respect to axiom A1, however, the w-index performs worse. The two vectors $x = (6, 6, 1, 1, 1)$ and $y = (6, 6, 3, 1, 1, 1)$ with $w(x) = 3 < 4 = w(y)$ illustrate that it violates axiom A1.

We have the following axiomatic characterization of the w-index.

Theorem 4.2. *A scientific impact index $f : X \to \mathbb{N}$ satisfies the axioms* A2, B*, and* C*, if and only if it is the w-index.*

Finally, the maximum-index $f_{\max}$ satisfies the four axioms A1, A2, C, and D, but violates axiom B. This is illustrated, for instance, by the two vectors $x = (1, 1, 1)$ and $y = (9, 1, 1)$ with $f_{\max}(x) = 1$ and $f_{\max}(y) = 9$. Here is an axiomatic characterization of the maximum-index.

Theorem 4.3. *A scientific impact index $f : X \to \mathbb{N}$ satisfies the axioms* A1, A2, *and* C, *if and only if it is the maximum-index* $f_{\max}$.

Let us reconsider our situation. We are working with a set of five axioms A1, A2, B, C, D. We know that there is no index that would simultaneously satisfy all five of them. But if we are willing to abandon one of these five axioms, then the above theorems show that sometimes we may find compatible impact indices:

+ The w-index is the unique index that satisfies all axioms except A1.
+ The Hirsch-index is the unique index that satisfies all axioms except A2.
+ The maximum-index is the unique index that satisfies all axioms except B.
– No index can satisfy all axioms except C.
– No index can satisfy all axioms except D.

The two negative statements follow from Theorem 3.2 (and the second negative statement also follows from Proposition 3.1). Furthermore, we claim that the three positive cases are tight in the following sense: The characterizations in Theorems 4.1–4.3 yield that in each positive case, the respective three characterizing axioms imply the fourth axiom. Theorem 4.4 shows that we cannot drop any of the characterizing axioms, without losing the uniqueness conclusion.

Theorem 4.4. *On the borderline to the* h*-index, there exist scientific impact indices that satisfy*

(a) *the axioms* B, C, D, *but not* A1*;*
(b) *the axioms* A1, C, D, *but not* B*;*
(c) *the axioms* A1, B, C, *but not* D.

On the borderline to the w*-index, there exist scientific impact indices that satisfy*

(d) *the axioms* B, C, D, *but not* A2*;*
(e) *the axioms* A2, C, D, *but not* B*;*
(f) *the axioms* A2, B, D, *but not* C.

On the borderline to the maximum-index, there exist scientific impact indices that satisfy

(g) *the axioms* A2, C, D, *but not* A1*;*
(h) *the axioms* A1, C, D, *but not* A2*;*
(i) *the axioms* A1, A2, D, *but not* C.

5. Proofs of the theorems

In this section we will prove all theorems formulated in the preceding sections.

We start with the proof of Theorem 3.2. Suppose for the sake of contradiction that there exists a scientific impact index f that satisfies all three axioms A1, A2, and B. We will first prove as an auxiliary result that $f(t) = 1$ holds for all integers $t \geq 1$. Definition 2.1 implies that the empty vector has index 0, and then axiom A2 yields $f(t) \geq 1$. Definition 2.1 implies that $f(0) = 0$, and then axiom B yields $f(t) \leq 1$. Putting things together, $f(t) = 1$ indeed holds for all integers $t \geq 1$.

This auxiliary result yields $f(1) = 1$, and then A2 implies $f(2, 1) > 1$. The auxiliary result also yields $f(2) = 1$, and then A1 and monotonicity imply $f(2, 1) = 1$. This contradiction completes the proof of Theorem 3.2. ■

We turn to the proof of Theorem 4.1. One direction of the proof is straightforward, since the Hirsch-index clearly satisfies the axioms A1, B, and D. For the other direction of the proof, we consider an arbitrary index f that satisfies axioms A1, B, and D. We will show that f is the h-index. Our argument proceeds in three steps.

In the first step, we argue that any vector x with at most k non-zero components has $f(x) \leq k$. This follows by an easy inductive argument, starting from Definition 2.1 and then repeatedly applying axiom B.

The second step considers for every $k \geq 0$ the vector $u^{[k]}$ that consists of exactly k components of value exactly k. We prove by induction on $k \geq 0$ that $f(u^{[k]}) = k$. The statement for $k = 0$ follows from

Definition 2.1. In the inductive step, we derive from the inductive assumption and from axiom D that $f(u^{[k+1]}) > f(u^{[k]}) = k$, whereas the statement in the first step yields $f(u^{[k+1]}) \le k+1$. This yields the desired $f(u^{[k+1]}) = k+1$.

In the third step we establish $f(x) \equiv h(x)$ for all x. Consider an arbitrary vector $x = (x_1, \ldots, x_n)$, and let $k := h(x)$. Let $y = (x_1, \ldots, x_k)$ denote the vector that consists of the first k components of x. Since these components all are at least k, we get $u^{[k]} \preceq y$. The monotonicity condition in Definition 2.1 implies $f(u^{[k]}) \le f(y)$. With this, the statement in the first step yields $f(y) \le k$ and the statement in the second step yields $f(y) \ge k$; hence $f(y) = k$. Since vector x results from vector y by adding components of values at most k, repeated application of axiom A1 and monotonicity gives $f(x) = f(y) = k$. Therefore $f(x) = h(x)$, and the proof is complete. ■

Now let us prove Theorem 4.2. Again, one direction of the proof is straightforward: The w-index satisfies the axioms A2, B, and C. For the other direction, we investigate an arbitrary index function f that satisfies axioms A2, B, and C. We will show that f coincides with the w-index. The argument goes through four steps.

The first step is identical to the first step in the proof of Theorem 4.1: Any vector x with at most k non-zero components satisfies $f(x) \le k$. The argument is an induction based on Definition 2.1 and axiom B.

In the second step we consider for every $k \ge 0$ the vector $v^{[k]} = (k, k-1, \ldots, 2, 1, 0)$. We prove by induction on $k \ge 0$ that $f(v^{[k]}) = k$: The case $k = 0$ is trivial. In the inductive step, the inductive assumption and axiom A2 together yield that $f(v^{[k+1]}) > f(v^{[k]}) = k$, and the statement in the first step yields $f(v^{[k+1]}) \le k+1$. We conclude that $f(v^{[k+1]}) = k+1$ holds indeed.

The third step proves the following statement by induction on $k \ge 0$: If a vector $x = (x_1, \ldots, x_n) \in X$ has some component x_m that satisfies $x_m \le k - m + 1$ for some $k \ge 0$, then $f(x) \le k$. For $k = 0$, only the case $m = 1$ is relevant. Then $x = (0, 0, \ldots, 0)$, and the statement follows from Definition 2.1. In the inductive step, we consider a vector x that has a component x_m with $x_m \le k - m + 2$. We consider two cases. In the first case $x_m = 0$ holds. Then $m \le k+2$, and vector x contains at most $k+1$ non-zero components. The first step yields the desired $f(x) \le k+1$. In the second case $x_m \ge 1$ holds. We decrease all non-zero components in x by 1, leave the zero components untouched, and denote the resulting vector $y = (y_1, \ldots, y_n)$. Then $y_m = x_m - 1 \le k - m + 1$, and the inductive assumption yields $f(y) \le k$. Axiom C gives $f(x) \le f(y) + 1 \le k + 1$.

In the fourth step we show $f(x) \equiv w(x)$. Consider an arbitrary $x = (x_1, \ldots, x_n)$, and let $k := w(x)$. Then $x_m \ge k - m + 1$ holds for all $m \le k$, and at least one of these inequalities is an equality. The equality case together with the third step result implies $f(x) \le k$. Since $v^{[k]} \preceq x$, the monotonicity condition and the second step imply $f(x) \ge k$. This yields $f(x) = k = w(x)$. The proof of Theorem 4.2 is complete. ■

Next, let us prove Theorem 4.3. Since the maximum-index $f_{\max}$ obviously satisfies the axioms A1, A2, and C, one direction of the proof is immediate. For the other direction, we consider an arbitrary index f that satisfies A1, A2, and C. We will show that $f \equiv f_{\max}$. The argument proceeds in four steps.

In the first step, we argue that $f(x) \le f_{\max}(x)$ holds for all $x \in X$. The proof can be done via straightforward induction, that starts from Definition 2.1 and then repeatedly applies axiom C.

The second step considers for every $k \ge 0$ the vector $v^{[k]} = (k, k-1, \ldots, 2, 1, 0)$. We prove by induction on $k \ge 0$ that $f(v^{[k]}) = k$: The statement for $k = 0$ follows from Definition 2.1. In the inductive step, we derive from the inductive assumption and from axiom A2 that $f(v^{[k+1]}) > f(v^{[k]}) = k$, whereas the statement in the first step yields $f(v^{[k+1]}) \le k+1$. This yields the desired $f(v^{[k+1]}) = k+1$.

In the third step we establish $f(k) \ge k$ for every integer $k \ge 0$. The proof is done by induction. The case $k = 0$ is trivial. In the inductive step, the monotonicity condition in Definition 2.1 together with the inductive assumption yields $f(k+1) \ge f(k) \ge k$. Suppose for the sake of contradiction that $f(k+1) = k$. If we add one by one the components $0, 1, 2, \ldots, k$ to the one-dimensional vector $(k+1)$, we end up with vector $v^{[k+1]}$. By axiom A1 these additions do not increase the index; hence $f(v^{[k+1]}) \le f(k+1) = k$ must hold. Since this blatantly contradicts the result derived in the second step, we may conclude the desired $f(k+1) \ge k+1$.

In the fourth step we establish $f(x) \equiv f_{\max}(x)$ for all x. Consider an arbitrary vector $x = (x_1, \ldots, x_n)$, and let $k := f_{\max}(x)$. The first step implies $f(x) \le k$. Since $(k) \preceq x$, the monotonicity condition and the third step imply $f(x) \ge k$. This shows $f(x) = k = f_{\max}(x)$, and completes the proof of Theorem 4.3. ■

Finally, we will prove Theorem 4.4. The w-index satisfies axioms A2, B, C, and D, but violates A1; this settles statements (a) and (g). The maximum-index satisfies axioms A1, A2, C, and D, but violates B; this settles statements (b) and (e). The h-index satisfies axioms A1, B, C, and D, but violates A2; this settles statements (d) and (h). For the remaining three statements (c), (f), and (i), we introduce the following indices.

- The zero-index assigns to every vector x the value 0. This trivial index satisfies the axioms A1, B, C, violates A2 and D, and thus settles statement (c).
- Next, consider the index that assigns to every vector the number of non-zero components. This index satisfies A2, B, D, violates A1 and C, and thus settles (f).
- Consider the scientific impact index that assigns to a vector $x = (x_1, \ldots, x_n)$ in X the value of the smallest even integer greater or equal to x_1. This index satisfies axioms A1, A2, D, but violates B and C. Therefore, it settles statement (i).

This completes the proof of Theorem 4.4. ■

6. Conclusions

In this article, we have given axiomatic characterizations for the Hirsch-index, the w-index, and the maximum-index. Out of these three indices, the maximum-index is clearly the least interesting. It is irrelevant for all practical purposes. Since it only measures the impact of a *single* publication, it cannot provide any useful information on the productivity or cumulative impact of a researcher.

The usefulness of the Hirsch-index, on the other hand, is widely recognized. It is transparent, unbiased and very hard to rig. Ball (2005) indicates that it might eventually be useful as an objective criterion for election to bodies such as the USA National Academy of Sciences or Britain's Royal Society. One weak point of the Hirsch-index is that it tends to cluster high numbers of scientists into the same index value.

The w-index is a contribution of the current article. It is a kind of dual to the Hirsch-index, and it fell out almost for free from our axiomatic investigations around the Hirsch-index. Similarly as the Hirsch-index, the w-index tends to cluster many scientists into the same index value. However, our Proposition 2.4 indicates that its range might be twice the range of the Hirsch-index. Therefore the w-index should lead to a somewhat finer ranking than the Hirsch-index. (We have verified this observation for a partial list of computer scienctists with high Hirsch-index according to *Google Scholar*.) Another nice property of the w-index is that its underlying isosceles right-angled triangle (see Fig. 1) seems to resemble one's average citation curve more closely than the square that underlies the Hirsch-index. All this makes us hope that the w-index might actually turn out to be useful for practical purposes.

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