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THE MULTIPLE JUDGE, MULTIPLE CRITERIA RANKING PROBLEM: A FUZZY SET APPROACH

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This paper investigates the problem of selecting, from a set of issues, those which best satisfy a collection of criteria. A group of judges have fuzzy sets defined over the issues, for each criterion, whose values lie in a finite linearly ordered set $\mathscr{L}$. These judges also have fuzzy sets defined over the set of criteria. The paper discusses methods of aggregating all the fuzzy sets into one fuzzy set μ, defined on the issues, so that $\mu(A) \in \mathscr{L}$ gives the final ranking for issue A.

Keywords: Multicriteria decision making, Social choice.

1. Introduction

Suppose we have m alternatives (issues, candidates, ...) called $A_1, A_2, \ldots, A_m$ to rank from 'best' to 'worst'. A study to accomplish this is designed by an analyst, or a team of analysts, who we will call 'AN' for short. The AN will employ the testimony of n judges (experts, ...) called $J_1, J_2, \ldots, J_n$. These judges are to supply information about the alternatives for each criterion (characteristic, ...) $C_1, C_2, \ldots, C_K$ and also information about the importance of the criteria with respect to some overall objective. The hierarchical structure is shown in Figure 1. The AN wishes to select from the $A_1, A_2, \ldots, A_m$ those which best satisfy the criteria.

For simplicity we are considering only one hierarchy, but our method may be extended to any number of hierarchies. If there is only one judge, then we have the traditional hierarchical structure. Saaty's [6], [7] method of hierarchical analysis employs a ratio scale. Our procedure, when restricted to the one judge case, produces a method of hierarchical analysis using an ordinal scale.

The AN designs a scale $\mathscr{L} = \{S_0, S_1, \ldots, S_L\}$ of preference information to be used by the experts. We assume that $\mathscr{L}$ is linearly ordered and $S_0 < S_1 < \cdots < S_L$. No other structure is assumed to exist on $\mathscr{L}$. The S_i are not numbers. For example, $\mathscr{L}$ could be $\{\emptyset, \mathrm{VL}, \mathrm{L}, \mathrm{M}, \mathrm{H}, \mathrm{VH}, \mathrm{P}\}$ where $\emptyset$ = none, VL = very low, L = low, M = medium, H = high, VH = very high, and P = perfect. In effect the judges will be using an ordinal scale and not an exact, ratio, or interval scale ([5], p. 64). Only ordinal information will be required from the experts.

expect all criteria to be rated low by the judges but still Yager's method gives issues high ratings when the criteria have low ratings. Consider another example. Suppose $m_{ik} = n_k = \mathrm{VH}$ for $2 \leq k \leq K$ and $m_{i1} = \emptyset$, $n_1 = \mathrm{P}$. Then $w_i = \emptyset$. This issue is severely penalized for one very low ranking. The problem now is that Q is the min operator.

As before we will impose majority rule on the aggregation function Q.

P5 (Majority rule). If, for a majority of criteria, $p_{ik} = S_t$, then $w_i = S_t$.

It follows from Theorem 1 that Q must be the median operator when the number of criteria is odd. But now K could be even. We would expect that quite often there are an odd number of judges. Surely we could have two, or four, criteria. The proof of Theorem 1 does show majority rule implies that quite often Q and the median operator must agree. In order to resolve the cases where the values of Q are still undetermined we will choose the median operator for Q.

We now need to determine the λ-table, or the values of the p_{ik}. At this point λ could be the max, min, or some type of mixed operator. We will require λ to have the following three properties:

(i) (symmetry) $\lambda(x, y) = \lambda(y, x)$,
(ii) $\lambda(x, x)$ is strictly increasing,
(iii) $\lambda(S_0, S_0) = S_0$ and $\lambda(S_L, S_L) = S_L$.

Since we are using the same scale $\mathscr{L}$ for both the m_{ik} and n_k it is natural to require symmetry. Also, if for some criterion C_k an issue receives the lowest possible ranking S_0 and that criterion also has the lowest possible weight, then when these are combined the result p_{ik} is the lowest possible ranking. Similarly, when the highest possible rankings S_L and S_L are combined, the result will be the highest possible ranking. Finally, if $x_1 = m_{is} = n_s > x_2 = m_{it} = n_t$ for some issue A_i, then we should give a higher weighted ranking to A_i for C_s than for C_t. That is, $\lambda(x_1, x_1) > \lambda(x_2, x_2)$.

It is easily seen that properties (ii) and (iii) imply that λ is **idempotent** in that $\lambda(x, x) = x$ for all $x \in \mathscr{L}$. Conditions (i), (ii), and (iii) are similar to some of the conditions used by Bellman and Giertz [1] in order to characterize max and min as those operators corresponding to the intersection and union of fuzzy sets.

A majority of the judges can determine any m_{ik} or n_k. We now ask that the same is true for the final rankings w_i. In the following property 0 is a subset of $\{1, 2, \ldots, K\}$ with $|0| > K/2$.

P10 (Citizen's sovereignty). If, for some issue A_i, a majority of the experts have $a_{ij}^k = S_t$ for all criteria C_k with $k \in 0$, then $w_i = S_t$.

In the above statement of citizen's sovereignty let $P = \{n_k \mid k \in 0\}$.

Theorem 2. *Citizen's sovereignty is possible if and only if* $|P| = 1$. *Let* $P = \{S_a\}$. *Then* λ *is the* max (min) *operator if and only if* $S_a = S_0$ ($S_a = S_L$). *Otherwise* λ *is a mixed operator.*

Proof. Suppose citizen's sovereignty holds and let $S_a \neq S_b \in P$. We show that $\lambda(S_i, S_a) = \lambda(S_i, S_b) = S_i$, $0 \leq i \leq L$, contradicting the symmetry of the λ-table.

Let $n_k = S_a$ for all $k \in 0$. A majority of the m_{ik}'s are equal to S_t. Therefore, a majority of the p_{ik}'s are equal and they must equal S_t if w_i is to equal S_t. Hence $\lambda(S_t, S_a) = S_t$. This must be true for all $S_t \in \mathscr{L}$. Similarly we obtain $\lambda(S_t, S_b) = S_t$ by setting $n_k = S_b$ for all $k \in 0$.

Now assume that $|P| = 1$ and let S_a be the only element in $\mathbf{P}$. Citizen's sovereignty results if a majority of the experts also agree that $b_{kj} = S_a$ for a majority of criteria. Of course we must have $\lambda(S_i, S_a) = S_i$, $0 \leq i \leq L$.

First let $S_a = S_0$. By symmetry we need only determine $\lambda(S_i, S_j)$ for $S_i \geq S_j$. Now $S_i = \lambda(S_i, S_0) \leq \lambda(S_i, S_j) \leq \lambda(S_i, S_i) = S_i$. Hence $\lambda(S_i, S_j) = \max(S_i, S_j) = S_i$.

Next let $S_a = S_L$. We only need to find $\lambda(S_i, S_j)$ for $S_i \leq S_j$. Now $S_i = \lambda(S_i, S_i) \leq \lambda(S_i, S_j) \leq \lambda(S_i, S_L) = S_i$. Therefore $\lambda(S_i, S_j) = \min(S_i, S_j) = S_i$.

Finally, assume that S_a does not equal S_0 or S_L. We show that

$$\lambda(S_i, S_j) = \begin{cases} \max(S_i, S_j) & \text{if } S_i, S_j \geq S_a, \\ \min(S_i, S_j) & \text{if } S_i, S_j \leq S_a. \end{cases}$$

This is what we call a mixed operator. Notice that the values of λ are not uniquely determined when $S_i > S_a$ and $S_j < S_a$ or $S_i < S_a$ and $S_j > S_a$. First consider the case where $S_i \leq S_j \leq S_a$. Then $S_i = \lambda(S_i, S_i) \leq \lambda(S_i, S_j) \leq \lambda(S_i, S_a) = S_i$. So $\lambda(S_i, S_j) = \min(S_i, S_j)$ for $S_i, S_j \leq S_a$. Next consider $S_a \leq S_j \leq S_i$. Then $S_i = \lambda(S_i, S_a) \leq \lambda(S_i, S_j) \leq \lambda(S_i, S_i) = S_i$. Therefore $\lambda(S_i, S_j) = \max(S_i, S_j)$ when $S_i, S_j \geq S_a$.

The max and min operators do not appear to be appropriate methods of combining rankings m_{ik} and criteria weights n_k. Therefore, we recommend the mixed operator for λ. Now we must decide on the value of S_a in Theorem 2 and the undetermined values of λ.

Let $l = L/2 + 1$ if L is even and $l = (L+1)/2$ if L is odd. Generally, a value of $S_j \in \mathscr{L}$ greater than or equal to S_l is considered a 'good' rating and a $S_j < S_l$ is a not good or a 'bad' rating. A majority of the weights n_k for the criteria should be in the good category. Otherwise, the AN should redesign the study and employ criteria better suited for the overall objective. Therefore, the S_a in Theorem 2 should be greater than S_l. For example, if $\mathscr{L} = \{\emptyset, \mathrm{VL}, \mathrm{L}, \mathrm{M}, \mathrm{H}, \mathrm{VH}, \mathrm{P}\}$, then S_a could be H or VH. Suppose $S_a = \mathrm{H}$. Then $\lambda(S_i, \mathrm{H}) = S_i$ for $0 \leq i \leq 6$ and the values of $\lambda(S_i, S_j)$ for $0 \leq i \leq 3$, $5 \leq j \leq 6$ are undetermined. The other values of $\lambda(S_i, S_j)$, $0 \leq j \leq 3$, $5 \leq i \leq 6$ are given by symmetry. We propose using the median operator to produce the undetermined values of λ. The resulting operator we will call MM(x, y) and is defined as follows:

$$\mathrm{MM}(x, y) = \begin{cases} \max(x, y) & \text{if } x, y \geq S_a, \\ \min(x, y) & \text{if } x, y \leq S_a, \\ \mathrm{Med}(x, y) & \text{otherwise,} \end{cases}$$

where $S_l \leq S_a < S_L$. The Med operator may be the round up $\overline{\mathrm{Med}}$ or the round down $\underline{\mathrm{Med}}$. Table 1 is the MM operator for $\mathscr{L} = \{\emptyset, \mathrm{VL}, \mathrm{L}, \mathrm{M}, \mathrm{H}, \mathrm{VH}, \mathrm{P}\}$ where $S_a = \mathrm{H}$.

Table 1. A λ-table using a mixed operator and a median (round-up) operator

	n_k						
	∅	VL	L	M	H	VH	P
m_{ik}	S_0	S_1	S_2	S_3	S_4	S_5	S_6
∅ $= S_0$	S_0	S_0	S_0	S_0	S_0	S_3	S_3
VL $= S_1$	S_0	S_1	S_1	S_1	S_1	S_3	S_4
L $= S_2$	S_0	S_1	S_2	S_2	S_2	S_4	S_4
M $= S_3$	S_0	S_1	S_2	S_3	S_3	S_4	S_5
H $= S_4$	S_0	S_1	S_2	S_3	S_4	S_5	S_6
VH $= S_5$	S_3	S_3	S_4	S_4	S_5	S_5	S_6
P $= S_6$	S_3	S_4	S_4	S_5	S_6	S_6	S_6

If we pool the experts first, then we have shown that it is reasonable to have the pooling functions F, G, and Q all equal to the median operator. Also, if citizen's sovereignty is desired, we have argued that a mixed operator in the form of $MM(S_i, S_j)$ is a good choice for λ.

4.2. *Pool last*

We would require the same basic properties (some minor rewording would be necessary for this case) of the λ and Q functions as when the judges were pooled first. Therefore, Q is the median operator and we recommend the mixed operator $MM(S_i, S_j)$ for λ.

Of course, even though the functions are the same whether we pool first or last, the final rankings can be different.

5. Summary and conclusions

In this paper we considered the problem of doing a study to select, from a set of issues $A_1, \ldots, A_m$, those which best satisfy a collection of criteria $C_1, \ldots, C_K$. To carry out this project we requested information from a group of judges $J_1, \ldots, J_n$ as to how well each issue satisfies each criterion and also how important each criterion is to the overall objective. We assumed that each judge has a fuzzy set defined over the issues, for each criterion, with values in some linearly ordered set $\mathscr{L}$. Also each judge has a fuzzy set defined over the criteria with values in $\mathscr{L}$. The problem is how to aggregate these fuzzy sets into one fuzzy set μ on the issues with values in $\mathscr{L}$ so that $\mu(A_i)$ is the final ranking for issue A_i.

In order to compute $\mu(A_i)$ we discussed the following three problems:

(1) when to pool the judges,

(2) how to pool the judges, and

(3) how to finally compute the values of $\mu(A_i)$.

We considered two ways to pool the experts: at the beginning or at the end. The major property imposed on the pooling functions was majority rule. We showed

that if n and k are odd, then the pooling functions must be the median operator. If n or k is even, then we argued that it was very reasonable to still use the median operator.

The method of combining an issue's ranking with a criterion's weight was accomplished by a function $\lambda : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$. The values of λ must be specified before $\mu(A_i)$ is known. The basic properties assumed to hold for λ were: (1) non-decreasing in both variables; (2) symmetric; and (3) idempotent. When we added the condition of citizen's sovereignty to the aggregation process, it was shown that λ must be a max, min, or mixed operator. We argued that a mixed operator of the form $MM(x, y)$, for max/min/median, was a good choice for λ.

The aggregation process then possessed the following important properties: (1) positive association of individual and group preference; (2) Pareto; (3) no judge or criterion can be dictatorial; (4) independence of irrelevant alternatives; (5) citizen's sovereignty.

The special case of one judge is the well-known hierarchical analysis problem studied by Saaty using a ratio scale. Our method then gives a hierarchical analysis procedure using fuzzy sets whose values lie in a finite linearly ordered set.

Another special case of one criterion has been called (fuzzy) multi-person decision making. See [2], Chapter 3, for a recent survey of this literature.

The conditions we placed on the pooling functions and λ all seemed quite natural. Researchers might wish to investigate other conditions to produce different methods of aggregating fuzzy sets. We need the least number of realistic conditions that will uniquely determine the pooling functions and λ producing an aggregation process with the maximum number of desirable properties.

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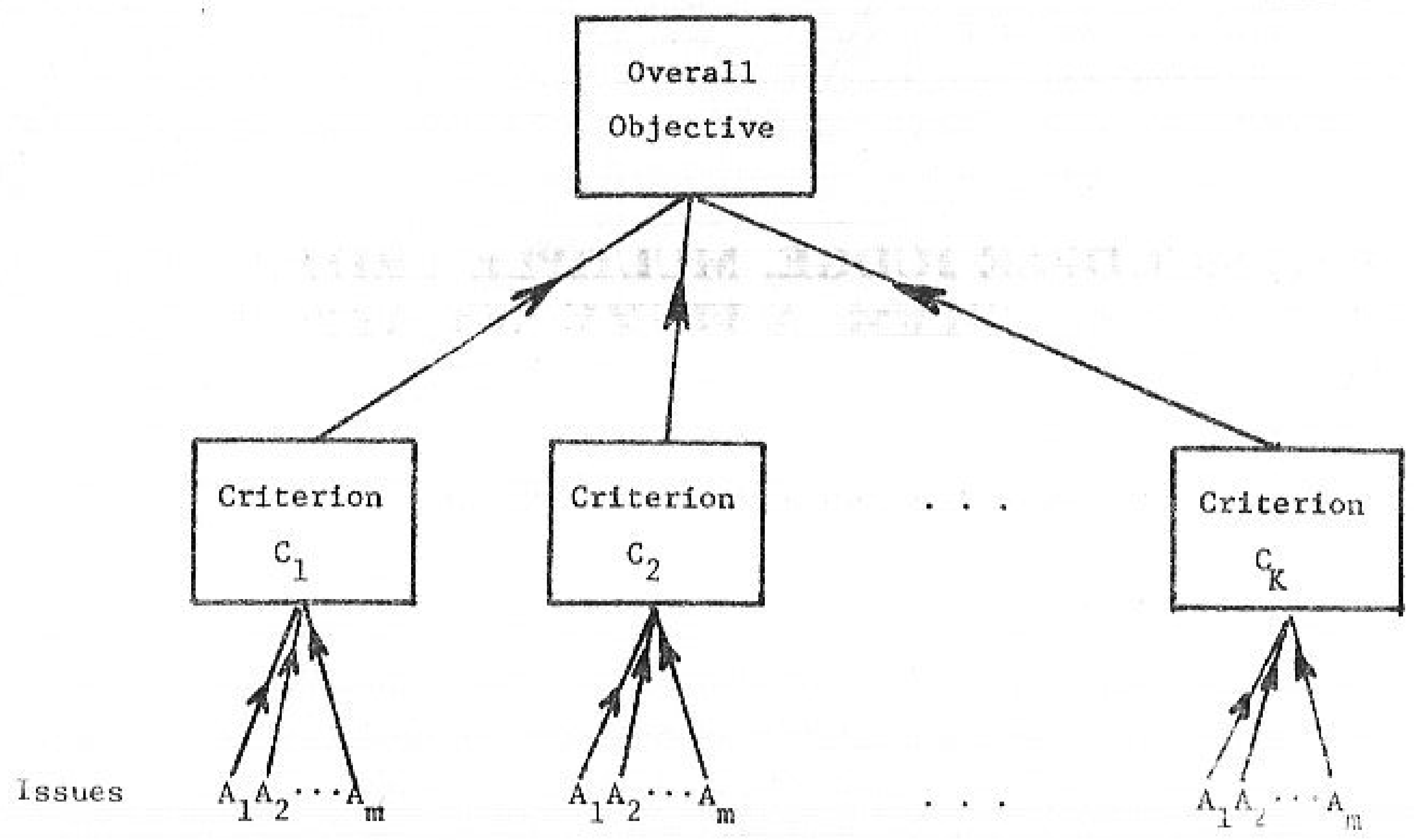

Fig. 1. Hierarchial structure.

There are various reasons for the AN to use an ordinal scale $\mathcal{L}$. It is probably easier for experts to assign $S_i \in \mathcal{L}$ to the alternatives and criteria than to assign numbers or ratios of numbers, especially when there are more than just a few alternatives and criteria. Also, some of the criteria may be vaguely understood or imprecisely defined for the judges. Then linguistic variables like 'low', 'high' are preferable. The evaluation process performed by the experts may be very subjective, and then it seems more appropriate to use an ordinal scale.

The judges assign an $S_i \in \mathcal{L}$ to the alternatives for each criterion and also to each criterion. Each judge J_j has a fuzzy set μ_j^k defined over the $A_1, A_2, \ldots, A_m$ with values in $\mathcal{L}$. Then $\mu_j^k(A_i)$ measures how well A_i satisfies C_k for judge J_j. Also, each judge J_j has a fuzzy set λ_j defined over the criteria $C_1, C_2, \ldots, C_K$ with values in $\mathcal{L}$. Then $\lambda_j(C_k)$ indicates the importance of criterion C_k with respect to the overall objective for judge J_j. We are using the same scale $\mathcal{L}$ for alternatives and criteria. With slight modifications our method could be extended to allow different ordinal scales for alternatives and criteria.

The data collected by the AN may be displayed in matrices T_k and T:

$$T_k = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} J_1 \; J_2 \; \cdots \; J_n \\ \left[\begin{array}{c} \\ \\ \mu_j^k(A_i) = a_{ij}^k \in \mathcal{L} \\ \\ \end{array} \right] \end{array},$$

for each criterion C_k, $1 \leq k \leq K$, and

$$T = \begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ C_K \end{array} \begin{array}{c} J_1 \; J_2 \; \cdots \; J_n \\ \left[\begin{array}{c} \\ \\ \lambda_j(C_k) = b_{kj} \in \mathcal{L} \\ \\ \end{array} \right] \end{array}.$$

Given the data T_k and T and AN now computes the final ranking of the issues given by $w = (w_1, w_2, \ldots, w_m)$ where $w_i \in \mathcal{L}$. Alternative A_i receives ranking w_i, $1 \leq i \leq m$. Again, we have assumed the same scale $\mathcal{L}$ for the final ranking of the issues. It is possible to have the w_i belong to a different linearly ordered set.

The final ranking produces disjoint sets $H_0, H_1, \ldots, H_L$, where some H_i could be empty, whose union is the set of issues and all alternatives in H_j have the same ranking $S_j \in \mathcal{L}$. If m is large and L small, then H_L could contain many alternatives. A second round of ranking for all the issues in H_L would be required in order to differentiate between these alternatives.

The problem outlined above is what we call the multiple expert, multiple criteria ranking problem. Three possible applications are:

1. *Grant proposals.* The grant proposals are the alternatives and the AN belongs to the agency awarding the grants. The experts are the people who review the grants. The scale $\mathcal{L}$ is usually numbers like $0, 1, 2, \ldots, 9$. Sometimes the AN ranks the criteria and the experts only supply the matrices T_k. Also, in some cases after the judges produce the T_k they are all brought together to somehow obtain the final ranking without ever ranking the criteria.

2. *Environmental hazards.* A government agency is asked to rank certain chemicals from most harmful to least harmful to the environment. The chemicals will be the alternatives. The criteria are various sections of the environment such as fish, wildlife, agriculture, timber, etc. The judges are scientists whose expertise is in this area. Exact (numbers) or ratio scales have been used when the number of chemicals is large.

3. *Energy development.* A government agency is asked to rank various alternatives from most important to least important with respect to energy development in the country over the next 10 years and over the next 25 years. The alternatives are nuclear power, wind power, solar power, etc., and the criteria might be cost, self sufficiency, etc. The judges are high ranking officials in energy related industry and government. The scale $\mathcal{L}$ would probably be numbers.

There are three problems that must be solved before the final ranking of the issues can be produced. They are: (1) when to pool, or average, the judges; (2) how to pool, or average, the judges; and (3) how to compute the final weights w_i. These problems are addressed in the next three sections.

2. When to pool

There seem to be two natural answers to this question: pool first, or pool last.

2.1. *Pool first*

If the AN first 'averages' across all the judges, then matrices T_k, $1 \leq k \leq K$, are used to compute matrix M, where

$$M = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} C_1 \quad C_2 \quad \cdots \quad C_K \\ \left[\begin{array}{c} \\ \\ m_{ik} \in \mathscr{L} \\ \\ \end{array} \right] \end{array},$$

and matrix T is used to produce matrix N, where

$$N = \begin{array}{c} C_1 \\ C_2 \\ \vdots \\ C_K \end{array} \left[\begin{array}{c} \vdots \\ \vdots \\ n_k \\ \vdots \end{array} \right], \qquad n_k \in \mathscr{L}.$$

The pooling, or averaging, procedure is accomplished by using functions

$$F: \prod_1^n \mathscr{L} \to \mathscr{L} \quad \text{and} \quad G: \prod_1^n \mathscr{L} \to \mathscr{L},$$

where

$$m_{ik} = F(a_{i1}^k, \ldots, a_{in}^k) \quad \text{and} \quad n_k = G(b_{k1}, \ldots, b_{kn}).$$

We will discuss properties of F and G in the next section. These functions combine the fuzzy sets of the judges and two candidates are max and min corresponding to the intersection and union of the fuzzy sets.

The kth column of M gives the ranking of the alternatives for criterion C_k across all the experts. Also, n_k in N is the 'weight' for criterion C_k obtained from all the judges. The computation of the w_i from M and N is now a one judge problem. This is discussed in the fourth section.

If the a_{ij}^k and b_{kj} are numbers, then it seems natural to average these numbers across all the judges to produce M and N. We could do this if we now assigned numbers to the $S_j \in \mathscr{L}$. This would be accomplished by two order-preserving mappings f and g from $\mathscr{L}$ into the real numbers. Apply f to the T_k and g to T to obtain matrices of numbers. We will not assign numbers to the $S_j \in \mathscr{L}$ because the final ranking (w_i) in general will depend on what order-preserving maps are used. Therefore, numbers should be used from the start in place of the linearly ordered set $\mathscr{L}$.

2.2. *Pool last*

The rankings w_{ij} for issue A_i for each judge J_j are computed first. The w_{ij} result from two functions

$$\lambda: \mathscr{L} \times \mathscr{L} \to \mathscr{L} \quad \text{and} \quad Q: \prod_1^K \mathscr{L} \to \mathscr{L},$$

where

$$q_{ij}^k = \lambda(a_{ij}^k, b_{kj}) \quad \text{and} \quad w_{ij} = Q(q_{ij}^1, q_{ij}^2, \ldots, q_{ij}^K).$$

The q_{ij}^k are the 'weighted' ranking of issue A_i for criterion C_k by judge J_j. The Q function combines these across all criteria for judge J_j.

The rankings w_{ij} are then pooled across all experts by use of the function

$$V: \prod_1^n \mathscr{L} \to \mathscr{L} \quad \text{where} \quad w_i = V(w_{i1}, w_{i2}, \ldots, w_{in}).$$

We will discuss properties of these functions in later sections. All the functions λ, Q, and V combine fuzzy sets and could be max, min, or some other operator. For example, λ and Q combine the fuzzy sets μ_j^k and λ_j for judge J_j into a fuzzy set Δ_j on $A_1, \ldots, A_m$ with values in $\mathscr{L}$ where

$$w_{ij} = \Delta_j(A_i).$$

The function V combines the fuzzy sets $\Delta_1, \Delta_2, \ldots, \Delta_n$ into one fuzzy set μ on $A_1, \ldots, A_m$ where

$$w_i = \mu(A_i).$$

3. How to pool

We first discuss properties of F and G.

3.1. *Pool first*

As a basic minimum we shall require that the pooling process have the following properties:

P1. If, for any issue A_i, some of the judges, for any criterion, raise their a_{ij}^k, then m_{ik} will not decrease. Similarly, if some experts raise their b_{kj} for any criterion C_k, then n_k will not decrease. This property might be called the positive association of individual and group preference.

P2. The m_{ik} and n_k do not change if the Judges are renumbered. That is, no judge can be a dictator.

It follows that F and G must have the following properties:
(i) Non-decreasing in each variable.
(ii) (Symmetric) $F(a_{i1}^k, \ldots, a_{in}^k)$ and $G(b_{k1}, \ldots, b_{kn})$ are unchanged if their arguments are permuted.

The pooling process will then have the following properties:

P3 (Pareto). If $a_{sj}^k \geq a_{tj}^k$, $1 \leq j \leq n$, then $m_{sk} \geq m_{tk}$. If $b_{sj} \geq b_{tj}$, $1 \leq j \leq n$, then $n_s \geq n_t$.

P4 (Independence of irrelevant alternatives and criteria). Suppose new issues $B_1, \ldots, B_r$ are added to the set of alternatives. If $m_{sk} \geq m_{tk}$ for the set of issues $A_1, \ldots, A_m$, then the same is true for the larger set of alternatives. If new criteria $D_1, \ldots, D_r$ are added to the set of criteria and $n_s \geq n_t$ for the set of criteria $C_1, \ldots, C_K$, then the same is true for the larger set of criteria.

There are many types of pooling functions F and G satisfying properties (i) and (ii) above. For example, F or G could be the max, min, mixed or median operator. A mixed operator is defined as follows:

$$\text{Mix}(x_1, \ldots, x_n) = \begin{cases} \min(x_1, \ldots, x_n) & \text{if all } x_i \geq S^*, \\ \max(x_1, \ldots, x_n) & \text{if all } x_i \leq S^*, \\ S^* & \text{otherwise,} \end{cases}$$

where S^* is any element in $\mathcal{L}$.

Fung and Fu ([3]; [4], p. 56) consider aggregating (pooling, averaging) operators on fuzzy sets and they show that if the operator satisfies certain properties it must be the max, min, or mixed operator when $\mathcal{L}$ is a connected topological space with the order topology. In their results $\mathcal{L}$ cannot be finite. In practice, the ordinal scale used by the judges will be finite. When $\mathcal{L}$ is finite there are other operators besides max, min and mixed which satisfy their properties.

The max and min operators do not seem to be appropriate for pooling, or averaging, experts. The mixed operator has the following undesirable property. Let $\mathcal{L} = \{\emptyset, \text{VL}, \text{L}, \text{M}, \text{H}, \text{VH}, \text{P}\}$ and set $S^* = M$ for the mixed operator. If all the judges, except one expert called J^*, assign P or VH to an issue A_i, and J^* assigns L or VL, then $m_{ik} = M$ for the mixed operator. Issue A_i is penalized by receiving one 'low' vote.

If the number of judges is odd, then the median operator $\text{Med}(x_1, \ldots, x_n)$ is defined for ordinal data. When n is even something must be done to break ties. If all the a_{ij}^k and b_{kj} are numbers, then averaging the a_{ij}^k and b_{kj} is a very reasonable method of pooling the experts. Since we cannot compute the numerical average of $S_i \in \mathcal{L}$, the median operator appears to be a good procedure of pooling the judges to produce matrices M and N.

We propose the following method of breaking ties when n is even. Suppose for $x_i \in \mathcal{L}$ the median of $x_1, \ldots, x_n$ lies between S_i and S_j in $\mathcal{L}$. Then $\text{Med}(x_1, \ldots, x_n) = S_t$ where $t = (i+j)/2$ if $i+j$ is even. When $i+j$ is odd we may round up or round down. That is, $\text{Med}(x_1, \ldots, x_n) = S_r$ for $r = (i+j+1)/2$ or $r = (i+j-1)/2$ when $i+j$ is odd. We will write $\overline{\text{Med}}$ when we always round up and $\underline{\text{Med}}$ when we always round down. For example, let $\mathcal{L} = \{\emptyset, \text{VL}, \text{L}, \text{M}, \text{H}, \text{VH}, \text{P}\}$ and assume that the judges assign VL, L, VH, VH, L, VH to some issue. Then the median is between $S_2 = \text{L}$ and $S_5 = \text{VH}$. Then $\overline{\text{Med}}$ produces H and $\underline{\text{Med}}$ gives M for this issue. When n is even, the median operator will be either $\overline{\text{Med}}$ or $\underline{\text{Med}}$.

If we require the pooling process to satisfy the following property, then F and G must be the median operator when the number of judges is odd. A majority of judges will be a simple majority. That is, if n is odd a majority is at least $(n+1)/2$ and if n is even a majority is at least $(n/2)+1$.

P5 (Majority rule). If for some issue A_i and criterion C_k a majority of the judges have $a_{ij}^k = S_t$, then $m_{ik} = S_t$. If for some criterion C_k a majority of the judges say $b_{kj} = S_t$, then $n_k = S_t$.

Theorem 1. (a) *Let n be odd. Majority rule holds if and only if F and G are the median operators.*

(b) *Let n be even. If F and G are the median operators, then majority rule holds. If majority rule holds, then F and G can be the median operator.*

Proof. (a) If F and G are the median operator, then clearly majority rule holds. Therefore, suppose that majority rule holds. We show that F must be the median operator. The proof for G is similar. We use the fact that F is non-decreasing in each variable and symmetric.

Given any $x_i \in \mathcal{L}$, we need to show $F(x_1, \ldots, x_n)$ is the median of $x_1, x_2, \ldots, x_n$. By symmetry we may assume there are integers $0 = r_0 < r_1 < r_2 < \cdots < r_s = n$ so that

$$\begin{aligned} x_i &= y_1 \quad \text{for } 1 \leq i \leq r_1, \\ x_i &= y_2 \quad \text{for } r_1 + 1 \leq i \leq r_2, \\ &\vdots \\ x_i &= y_s \quad \text{for } r_{s-1} + 1 \leq i \leq n, \end{aligned}$$

where $y_1 < y_2 < \cdots < y_s$. If any $r_i - r_{i-1} \geq (n+1)/2$, then $F(x_1, \ldots, x_n) = y_i$ which is the median of $x_1, \ldots, x_n$. So assume $r_i - r_{i-1} < (n+1)/2$, $i = 1, 2, \ldots, s$. Let $t = (n+1)/2$. We show that $F(x_1, \ldots, x_n) = x_t$ which is the median of $x_1, x_2, \ldots, x_n$. Let

$$\bar{x} = (x_t, x_t, \ldots, x_t, x_{t+1}, \ldots, x_n) \quad \text{and} \quad \underline{x} = (x_1, x_2, \ldots, x_{t-1}, x_t, x_t, \ldots, x_t).$$

In $\bar{x}$ we have increased x_i, $1 \leq i < t$, up to x_t and left all the other x_i unchanged. In $\underline{x}$ we have decreased x_i, $t < i \leq n$, down to x_t and left all the other x_i unchanged. Now

$$F(\underline{x}) \leq F(x_1, \ldots, x_n) \leq F(\bar{x}).$$

But both $F(\underline{x})$ and $F(\bar{x})$ equal x_t because a majority of the x_i equal x_t. Hence $F(x_1 \cdots x_n) = x_t$ also.

(b) Clearly, if F and G are the median operator, then majority rule holds. Therefore, assume that majority rule holds. Many, but not all, values of F and G are determined because of majority rule.

Let us consider the values of $F(x_1, \ldots, x_n)$ for any $x_i \in \mathcal{L}$. We employ the same notation as in part (a) above. Let $u = n/2$ and $v = u + 1$. The only values of F undetermined by majority rule is when $x_u = y_i$ for some i and $x_v = y_{i+1}$. Then

$x_u \leq F(x_1, \ldots, x_n) \leq x_v$. Of course, the values of F are not always completely arbitrary between x_u and x_v but will be somewhat determined by the fact that F is nondecreasing in each variable.

First suppose that $x_u = x_v = y_i$ for some i. Then we show that $F(x_1, \ldots, x_n) = x_u$ = the median of $x_1, \ldots, x_n$. Let $\bar{x}$ be constructed from $x_1, x_2, \ldots, x_n$ by increasing all x_i, $1 \leq i < u$, up to x_u and leaving all the other x_i unchanged. Let $\underline{x}$ be obtained from $x_1, x_2, \ldots, x_n$ by decreasing all x_i, $v < i \leq n$, down to x_u and leaving the other x_i unchanged. Then $F(\underline{x}) \leq F(x_1, \ldots, x_n) \leq F(\bar{x})$. But $F(\underline{x})$ and $F(\bar{x})$ both equal x_u.

Next assume that $x_u \neq x_v$ and the only possibility is for $x_u = y_i$ for some i and $x_v = y_{i+1}$. Construct $\bar{x}$ from $x_1, x_2, \ldots, x_n$ by increasing all x_i, $r_{i-1}+1 \leq i \leq r_i$, up to x_v and not changing any other x_i. Also let $\underline{x}$ be obtained from $x_1, x_2, \ldots, x_n$ by decreasing all x_i, $r_i+1 \leq i \leq r_{i+1}$, to x_u and leaving all the other x_i unchanged. Then

$$x_u = F(\underline{x}) \leq F(x_1, \ldots, x_n) \leq F(\bar{x}) = x_v.$$

The only other condition on F is that it is nondecreasing in each variable.

The max, min, and mixed operators do not satisfy majority rule. We will impose majority rule on the pooling process and therefore we will choose the median operator for F and G. When n is even the median operator is not an unreasonable method of pooling the judges. Suppose $\mathscr{L} = \{\emptyset, \text{VL}, \text{L}, \text{M}, \text{H}, \text{VH}, \text{P}\}$ and six judges assign $\emptyset$, VL, VL, H, H, and VH. If $x = (\emptyset, \text{VL}, \text{VL}, \text{H}, \text{H}, \text{VH})$, then the majority rule implies that $\text{VL} \leq F(x) \leq \text{H}$. The median operator computes $\overline{\text{Med}}(x) = \text{M}$, or $\underline{\text{Med}}(x) = \text{L}$.

3.2. *Pool last*

We will require the pooling, or averaging, method to satisfy majority rule.

P5 (Majority rule). If, for some issue A_i, a majority of the judges have $w_{ij} = S_t$, then $w_i = S_t$.

Theorem 1 implies that V must be the median operator when n is odd. Therefore, we will choose the median operator for V.

4. Computing the final weights

Again we consider two cases of pooling first or pooling last.

4.1. *Pool first*

We first need to combine the m_{ik} and n_k to obtain the weighted ranking for each issue and each criterion. Let

$$\lambda : \mathscr{S} \times \mathscr{L} \to \mathscr{L},$$

and define

$$p_{ik} = \lambda(m_{ik}, n_k).$$

The p_{ik} are the result of combining a criterion's weight n_k and an issues ranking for that criterion. Next we need to pool, or average, across all criteria. Let

$$Q : \prod_1^K \mathscr{L} \to \mathscr{L},$$

and define

$$w_i = Q(p_{i1}, p_{i2}, \ldots, p_{iK}).$$

The function Q aggregates across all criteria to obtain the final ranking w_i for issue A_i. At this point functions λ and Q need not be the same as those defined in Section 2 for the procedure of pooling last.

As a minimum we require the method of computing the w_i to possess the following properties:

P6. If, for some issue A_i and criterion C_k, some judges increase their a_{ij}^k and n_k remains unchanged, then w_i will not decrease. If, for some criterion C_k, some experts raise their b_{kj} but do not change their a_{ij}^k, then w_i will not decrease.

P7. The w_i do not change if the criteria are renumbered.

Therefore, λ and Q will have the following properties:
(i) Q is non-decreasing in each variable,
(ii) Q is symmetric,
(iii) λ is non-decreasing in each variable.
It follows that the ranking method has the following properties:

P8 (Pareto). If $a_{sj}^k \geq a_{tj}^k$ for all j and k, then $w_s \geq w_t$.

P9 (Independence of irrelevant alternatives). Suppose new issues $B_1, \ldots, B_r$ are added to the set of alternatives. If $w_i \geq w_j$ for the set of issues $A_1, \ldots, A_m$, then the same is true for the larger set of issues.

There are many pairs of functions λ and Q that might be employed to compute the w_i. Yager ([10], see also [8], [9]) proposed

$$w_i = \min_k (\max(m_{ik}, n_k')),$$

where $n_k' = S_{L-i}$ if $n_k = S_i \in \mathscr{L}$. That is, Q is the min operator and

$$\lambda(m_{ik}, n_k) = \max(m_{ik}, n_k').$$

Yager's λ function is non-increasing in its second variable.

Yager's method of computing w_i has the following undesirable properties. Let $\mathscr{L} = \{\emptyset, \text{VL}, \text{L}, \text{M}, \text{H}, \text{VH}, \text{P}\}$. Let $n_k = L$ for all k and suppose $m_{sk} = \emptyset$ for all k and $m_{tk} = H$ for all k. Then both issues receive a ranking of H. We would not