Decision Sciences, 12: 589-600, 1981

A NEW METHODOLOGY FOR ORDINAL MULTIOBJECTIVE DECISIONS BASED ON FUZZY SETS

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ABSTRACT

Using the structural forms supplied by fuzzy set theory and approximate reasoning, a new method is presented for solving multiple-objective decision problems for which the decision maker can supply only ordinal information on his preferences and the importance of the individual objectives.

Subject Areas: Decision Processes, Fuzzy Sets, and Group Decision Processes.

INTRODUCTION

Multiple-objective decision making involves interesting and complex issues that have stimulated considerable research aimed at their resolution (see, e.g., [4], [8], [15], and [27]). Of particular interest, Bellman and Zadeh's [3] paper on decision making in a fuzzy environment spawned considerable research into the application of fuzzy subsets to multiobjective decision making. Using a linear programming approach, for example, Zimmermann [25] [26] extended the Bellman-Zadeh model to include continuous variables, while Yager [16] [17] extended the model to allow for objectives of differing importances and subsequently [18] showed the connection between the fuzzy set approach and Zeleny's displaced ideal concept [23] [24] of solving multiobjective problems. Orlovsky [11], DuBois and Prade [5], Baas and Kwakernack [1], and Jain [6] have also worked on multiobjective problems using fuzzy subsets.

Two principal problems and certainly key issues in solving a multiobjective decision problem are to obtain meaningful information regarding the satisfaction of the objectives by the alternatives and to rank or weight the relative importance of the objectives.

This paper presents a procedure for multiobjective decision making with objectives that differ in importance to the decision maker. The procedure, which is based upon the Bellman-Zadeh approach, extends my previous work [16] [17] insofar as it represents a decision calculus that requires only *ordinal* information on preferences and importances.

MULTIOBJECTIVE DECISION MAKING

Multiobjective decision problems commonly require the choice of one element from a set $\{X\}$ of possible alternatives given a finite collection $\{A\}$ of criteria or objectives of concern to the decision maker. When $\{X\}$ is also finite, two significant and, as we shall see, related problems arise: evaluating how well each alternative satisfies the various objectives and combining the objectives to form an overall objective or decision function D from which the best alternative is to be selected. A third problem, one that is not considered in this paper, arises when $\{X\}$ is very large or infinite. Then the problem of efficiently evaluating the alternative set generally requires some type of calculus or mathematical programming solution.

Evaluating alternatives with respect to objectives requires extracting from decision makers their possibly subjective assessments of this information. The greater the precision required of the assessments, the greater is the difficulty in extracting them from the decision maker and the more likely it is that they will be inaccurate. The least informative assessments, linear orderings over each objective, are the easiest to obtain. Interval, ratio, and absolute evaluations are progressively more difficult to obtain. Furthermore, as the assessment scale becomes more refined it becomes more sensitive to "noise" and, consequently, more error prone. For example, although it may be straightforward to express a preference for x over y for some purposes, if one says x has a value of 2 and y has a value of 1.6 for those purposes, a slight change of the latter value to 1.7 may affect a final decision involving x and y. Yet, it may be difficult for the decision maker to make the fine distinction between 1.6 and 1.7 or to know what such a distinction implies.

The problem of combining objectives can be considered to involve two aspects resulting in an overall decision function D, which is a mapping from the set of alternatives X into some set S which has at least a linear ordering on its elements. In particular, $D(x)=f[A_1(x), A_2(x), \ldots, A_p(x)]$ for each $x \in X$, where $A_i(x)$ indicates the satisfaction of alternative x to objective A_i.

Consider, for example, the problem of selecting a car from a set $\{X\}$ of potential cars. The criteria $\{A\}$ that our decision maker may want to consider are A_1 = cost, A_2 = gas mileage, A_3 = comfort, and A_4 = repair frequency. The first aspect in constructing the decision function consists of specifying the relationships between the objectives in the formulation of the overall decision function D. That is, the decision maker may desire a car that satisfies all these objectives, but he may be willing to reconcile himself to accepting trade-offs between them. For example, if a car is very mileage efficient, he may be willing to forego some comfort; or, if the car is very repair free and mileage efficient, he may be willing to pay more for the car. These types of trade-offs can alternatively be reflected in importances associated with the objectives [18].

The second aspect associated with the construction of the decision function involves converting the decision rules of the type stated above into tractable "mathematical" operations meaningfully performable on the individual objective functions. It is at this point we interact with evaluation procedure. As is well known in the theory of measurement [13] [21], the scale on which we have our evaluations determines the class of operations available to be used to represent the relationships between the individual objectives in the overall decision function. As we go from an ordinal to an interval to a ratio and eventually to an absolute scale to measure preferences within the various objectives, we are in turn increasing the options available to us to represent the relationships between the

This model gives us a Pareto optimal solution. In particular, for any two alternatives x and y, if $A_i(x) \geq A_i(y)$ for all i, then $D(x) \geq D(y)$.

To see this we note first that $D(x) = \underset{i}{\text{Min}}[C_i(x)]$ and $D(y) = \underset{i}{\text{Min}}[C_i(y)]$ and if $C_i(x) \geq C_i(y)$ for all i then $D(x) \geq D(y)$.

Furthermore, since $C_i(x) = b_i' \vee A_i(x)$ and $C_i(y) = b_i' \vee A_i(y)$ then $A_i(x) \geq A_i(y)$ implies $C_i(x) \geq C_i(y)$. These two facts prove our observation.

Second, our solution is independent of irrelevant alternatives. That is, if we are given some set of alternatives X and find that our optimal solution is $x^* \in X$ and then if we consider some extended set of alternatives $Y = X \cup Z$, our optimal solution will either be x^* or some member of Z; never some other member of X.

To see this we note that if $D(x)$ is the value of x in our decision function when considering the set X, and if $E(x)$ is the value of x in our decision function when considering the alternate set Y, $D(x) = E(x)$ for all $x \in X$, and hence the $x \in X$ that is maximal over X is the same when considering X and Y. The only possible source of an alternative greater than x^* is one in Z.

Finally, the model also has the property that the more important is an objective, the more significant its effect on D. [Received: April 1, 1980. Accepted: January 23, 1981.]

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objectives. If our preference scales are too weak, we may not be able to find the mathematical operations necessary to describe in a meaningful way the relationships between the objectives. Thus, a trade-off has to be made in the solution of multiobjective decision functions. If we require preference information in sufficient detail to construct a scale necessary to enable us to perform the operations needed to describe fully the detailed relationships between the objectives, we may be forcing the decision maker to supply information with greater precision than he is capable of providing. This may lead to incorrect answers effected by errors in the supposedly precise information imparted into the decision process. If, however, we allow the decision maker to supply the information in a form with which he is confortable, we may not be provided with enough structure to describe the relationship among the objectives adequately. In this case we may get the wrong answer due to a lack of ability to describe D correctly. In the ideal situation we would have a model that has the facility of describing complex relationships that incorporate information from a scale with respect to which the decision maker can readily provide preferences.

FUZZY SET APPROACH

Bellman and Zadeh [3] have suggested an approach to multiobjective decision making based upon fuzzy subsets.

First, they suggest that each objective can be represented as a fuzzy subset over the set of alternatives $\{X\}$. Thus, if A_i indicates the ith objective, then the grade of membership of alternative x in A_i, $A_i(x)$, indicates the degree to which x satisfies the criteria specified by this objective.

The second contribution of their approach is to suggest a manner in which the objectives are related to form the decision function D. In particular, they suggested that when one has a collection of objectives to be satisfied, the overall objective is $D = A_1$ and A_2 and A_3 . . . and A_p. That is, one is seeking solutions that satisfy A_1 and A_2 and A_3 and A_4, etc.

Bellman and Zadeh [3] then suggest that, absent other information, the appropriate mathematical form for the "and" operation is the min operator that was originally suggested for the intersection of fuzzy subsets by Zadeh in his seminal paper [22]. Thus, the decision function is

$$D = A_1 \cap A_2 \cap A_3 \cap \ \ldots \ \cap A_p \tag{1}$$

where for each $x \in X$

$$D(x) = \mathrm{Min}[A_1(x), A_2(x), \ldots, A_p(x)]. \tag{2}$$

Finally, the optimal solution x^* satisfies

$$D(x^*) = \underset{x \in X}{\mathrm{Max}}\ D(x). \tag{3}$$

The Bellman-Zadeh approach always leads to a Pareto optimal solution. More generally, it is easy to show that if $A_i(x) \geq A_i(y)$ for all i, then $D(x) \geq D(y)$.

With respect to the discussion in the previous section, the requisite calculations for this approach do not require a very fine preference scale. In particular, if $\{S\}$ is a set of elements that has the structure of an ordinal scale, then the grades of membership of the fuzzy subsets can be chosen from this set and we can work out the Bellman-Zadeh optimal. Examples of this type of scale would be:

1. $S = \{$perfect, very high, high, medium, low, very low, none$\}$
2. $S = \{10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$
3. $S = \{$the unit interval$\}$

Thus the Bellman-Zadeh approach requires information from the decision maker only in terms of a linear ordering. Informationally, this approach is not very demanding; it does not, however, permit one to distinguish between the importance of the objectives. Thus we are very limited in the relationships available.

In [16] and [17], I have suggested an extension of this approach to allow for the inclusion of the importance of the various objectives. In particular, I have suggested that we can associate with each objective a value α_i indicative of its importance and then calculate

$$D = A_1^{\alpha_1} \cap A_2^{\alpha_2} \cap A_3^{\alpha_3} \cap \ \ldots \ \cap A_p^{\alpha_p} \tag{4}$$

where for each $x \in X$

$$D(x) = \mathrm{Min}[(A_1(x))^{\alpha_1}, (A_2(x))^{\alpha_2}, \ldots, (A_p(x))^{\alpha_p}]. \tag{5}$$

While this approach has extended the available relationships between the objectives, it has also put additional restrictions on the type of membership information necessary to carry out the required additional operations in a meaningful way. First, in using this method the grades of membership must be in the form of numbers, usually from the unit interval, and the αs, the importance measures that are associated with the objectives, must be nonnegative numbers. From a scale point of view, this approach requires that the grades of membership be drawn from an absolute scale while the importance measures may be drawn from a ratio scale. This can be seen as follows:

$$\mathrm{Min}[a^{k\alpha_1}, b^{k\alpha_2}] = \mathrm{Min}[(a^{\alpha_1})^k, (b^{\alpha_2})^k] = (\mathrm{Min}[a^{\alpha_1}, b^{\alpha_2}])^k. \tag{6}$$

One implication of this result is that the importance measures can be expressed on the unit interval, just as the membership grades.

Second, then, the information supplied to the analyst by the decision maker must be in the form of membership grade measures that are of absolute significance and importance measures that reflect "how many times more important" one objective is than a second. In some cases it may be very difficult for a decision maker to supply information in this detail. We shall try to develop a procedure that will give us the more varied ability to describe the relationship between

the objectives than does my earlier extension but that, like the Bellman-Zadeh formulation, requires little detail in specification of membership grades.

SET THEORY AND LOGIC

In anticipation of presenting our new methodology for multiobjective decision making we must lay some foundations in logic.

Just as there exists an intimate relationship between set theory and two-valued predicate logic, a similar relationship exists between multivalued logic and fuzzy set theory. In addition there exists some relationship between two-valued and multivalued logics.

In two-valued logic we have a set $\{S\}$ consisting of two elements having the ordering $1>0$ and some operations on these elements.

1. The meet, denoted $\wedge$, defined by
   $1\wedge 1=1;\ 1\wedge 0=0;\ 0\wedge 1=0;\ 0\wedge 0=0.$
2. The join, denoted $\vee$, defined by
   $1\vee 1=1;\ 1\vee 0=1;\ 0\vee 1=0;\ 0\vee 0=0.$
3. Negation, denoted $'$, defined by
   $1'=0;\ 0'=1.$
4. Implication, denoted $\rightarrow$, defined as follows:
   $1\rightarrow 0=0;\ 1\rightarrow 1=1;\ 0\rightarrow 0=1;\ 0\rightarrow 1=1.$

We can express this implication operation in terms of the other operations as follows: If $a,\ b\in S$, then $a\rightarrow b=a'\vee b$.

We can relate these operations to ordinary set theory. Assume A and B are two subsets from some universe $\{U\}$ containing all the elements under consideration. We can look at $\{A\}$ as a subset of $\{U\}$ where $A(u)$ indicates the grade of membership of u in A. In particular, if $u\in A$, then $A(u)=1$, and if $u\notin A$, $A(u)=0$. An ordinary subset is a fuzzy subset with membership drawn from $\{0,1\}$. Similarly, we could describe $\{B\}$. We can then express the operations of set theory in terms of operations in two-valued logic. If $C=A\cap B$, then the membership function of $\{C\}$ is defined such that, for each $u\in U$, $C(u)=A(u)\wedge B(u)$. Similarly, if $D=A\cup B$, then $D(u)=A(u)\vee B(u)$. Also, if $E=A'$, then $E(u)=A'(u)$. Finally, if $F=A\rightarrow B$, then $F(u)=A(u)\rightarrow B(u)$.

In fuzzy set theory, instead of the underlying membership set being a two-valued set it is a multivalued set, $\{L\}$, that has the structure of a lattice with a minimal and maximal element that we will still denote as 0 and 1. Furthermore, if $\wedge$, $\vee$, $\rightarrow$, and $'$ are all defined on this set $\{L\}$, then we can use these operations to define, as in ordinary set theory, operations on fuzzy subsets.

The only question left is the determination of the operations $\vee$, $\wedge$, $\rightarrow$, and $'$ on $\{L\}$. This involves the extension of two-valued logic operations to multivalued logic. This problem is discussed in detail in [12]. The particular form of the extension of the operation from two-valued logic to multivalued logic depends upon the set $\{L\}$ used to carry the multivalued logic. Two important cases of $\{L\}$ are of interest: when $\{L\}$ is a finite simply ordered set, we shall denote it as $\{H\}$; and, when $\{L\}$ is the unit interval, we shall denote it as $\{I\}$.

For any $\{L\}$ we can extend the meet operation as follows for $a,\ b\in L$, $a\wedge b=\text{Min}[a,\ b]$.

For any $\{L\}$ we can extend the joint operation, for $a, b\in L$, $a\vee b=\text{Max}[a,\ b]$.

As Lowen [9] has suggested, the multiple-value-logic extension of negation should have two properties:

1. For all $a,\ b\in L$, if $a\geq b$ then $a'\leq b'$; this is called order reversal.
2. For all $a\in L$, $(a')'=a$; this is called involution.

For $L=I$, an operation that is generally used as the negation is $a'=1-a$.

For $L=H$, we can define negation as follows: recalling $\{H\}$ is a finite simply ordered set, we can denote the elements as $H=\{h_0,\ h_1,\ h_2,\ \ldots,\ h_n\}$ where $i>j\Rightarrow h_i>h_j$. Then, we define $h_i'=h_{n-i}$.

The question of the appropriate extension of the implication operation has generated much literature [2] [10] [14] [20]. Many operations have been suggested, particularly when $L=I$. Two particular possibilities in this case are of interest. The first is based upon the relationship that exists between implication, negation, and join in two-valued logic: that is, $a\rightarrow b=a'\vee b$. The second has been recently suggested by Yager [20], $a\rightarrow b=b^a$. Whereas b^a can only be used when $L=I$, $a'\vee b$ can also be used when $L=H$.

A MODEL USING ORDINAL INFORMATION

As previously mentioned, the Bellman-Zadeh approach to multiobjective decision making has the advantage of requiring only an ordinal evaluation of the preference information but the disadvantage of not allowing one to include the fact that the objectives differ in importance. The method suggested by Yager is just the opposite: it allows one to capture differing importances between objectives, but it requires stronger information on preferences. In this section I shall suggest a model that has the advantages of both. This model will allow one to include the differing importance factor while still only requiring an ordinal scale for preference information.

Assume that $\{S\}$ is the finite set of elements used to indicate preference information. Furthermore, assume that the only structure available on $\{S\}$ is a linear ordering. Let $Y=\{A_1, A_2, \ldots, A_p\}$ be the set of criteria to be satisfied and let $\{X\}$ be our set of alternatives. Assume that each objective is represented by a fuzzy subset of X with grades selected from S. Thus, for any $x\in X$, $A_i(x)\in S$ indicates the degree to which x satisfies the criteria specified by A_i. Let G be a fuzzy subset of Y in which $G(A_i)\in S$ indicates the importance of the objective A_i. We shall denote $G(A_i)=b_i\in S$. Thus we have for each objective a measure of how important it is to the decision maker for this decision.

Based upon the approach suggested by Yager, and other multiobjective methods that include importance [4] [27] [8] [15], we conjecture a general form for this type of decision function,

$$D(x)=M(A_1(x),\ b_1) \text{ and } M(A_2(x),\ b_2)\ \ldots \text{ and } M(A_p(x),\ b_p), \tag{7}$$

where $M(A_i(x), b_i)$ indicates objective A_i evaluated at alternative x, modified by its importance. In [16] and [17], it is suggested that $M(A_i(x), b_i) = (A_i(x))^{b_i}$. As I have also indicated, however, this type of operation is not available to us when $A_i(x)$ and b_i are drawn from the finite linearly ordered set $\{S\}$. We must find some operation to replace this exponentiation. In our discussion of the implication operation we have shown that a^b and $b' \vee a$ are both acceptable operations for implication. That is, they both generally act in the same manner. Whereas a^b requires more than an ordinal scale to implement, however, $b' \vee a$ needs only a finite linearly ordered set on which the appropriate negation can be defined. This leads us to conjecture that in S, $M(A_i(x), b_i) = b_i' \vee A_i(x)$. Thus we are left with the conclusion that an appropriate model for including importances when our preferences are in S is

$$D = (b_1' \cup A_1) \cap (b_2' \cup A_2) \cap \ldots \cap (b_p' \cup A_p)$$

$$D = \bigcap_{i=1}^{p} (b_i' \cup A_i) \tag{8}$$

where $b_i' \cup A_i = C_i$ is a fuzzy subset of X defined as follows: $C_i(x) = b_i' \vee A_i(x)$. The optimal alternative is the $x \in X$ that maximizes D.

Let us examine this model to assure its not being counterintuitive.

First,

$$D = C_1 \cap C_2 \cap C_3 \cap \ldots \cap C_p \tag{9}$$

where $D(x) = \mathrm{Min}[C_1(x), C_2(x), \ldots, C_p(x)]$. The implication of this is that the representative of x in D is selected as the $C_i(x)$ that has the smallest value. Thus, $\underset{i}{\mathrm{Min}}\, C_i(x)$ becomes the most significant element in determining x's contribution to the overall decision function D. We would hope that as an objective becomes more important it plays a more significant role in determining D. Recalling that $C_i(x) = b_i' \vee A_i(x) = \mathrm{Max}[b_i', A_i(x)]$, consider the case when A_i is the least important objective; that is, $b_i = 0$, the minimal element of S. Since negation is order revising, this implies $b_i' = 1$ and hence $\mathrm{Max}[1, A_i(x)] = 1 = C_i(x)$. Since $D(x) = \underset{i}{\mathrm{Min}}\, C_i(x)$, it is very unlikely that $C_i(x)$ will be the determining value of $D(x)$. More, generally as the ith objective becomes more important, b_i increases, causing b_i' to get smaller which in turn causes $\mathrm{Max}[b_i', A_i(x)] = C_i(x)$ to be decreasing and also increases the likelihood that $C_i(x) = A_i(x)$. Since $D(x) = \underset{i}{\mathrm{Min}}\, C_i(x)$, it increases the possibility of $D(x)$ being determined by $A_i(x)$, the grade of membership of the most important objective. Furthermore, since the optimal alternative is x^* s.t.

$$D(x^*) = \underset{x \in X}{\mathrm{Max}}\, D(x), \tag{10}$$

we see that for a given $y \in X$, if $A_i(y)$ is low in the more important objective, it is unlikely that y will be selected as the optimal solution. We can thus see that the proposed model satisfies our intuitive requirements and allows us to include importance measures while still defined by operations that are performable on a linearly ordered finite set L.

EXAMPLE

Assume we must select a car from the set $X = \{\text{Chevrolet, Buick, Toyota}\}$, given the criteria set $A = \{\text{comfort, cost, gas mileage, repair frequency}\}$. The set S measures preferences, where $S = \{\text{lowest or none, very low, low, medium, high, very high, perfect or absolute}\}$. First, we rate the cars with respect to the objectives:

$$\text{comfort} \quad := A_1 = \left\{ \frac{\text{Medium}}{\text{Chevrolet}}, \frac{\text{Very High}}{\text{Buick}}, \frac{\text{Low}}{\text{Toyota}} \right\}$$

$$\text{cost} \quad = A_2 = \left\{ \frac{\text{Medium}}{\text{Chevrolet}}, \frac{\text{Very Low}}{\text{Buick}}, \frac{\text{High}}{\text{Toyota}} \right\}$$

$$\text{gas mileage} \quad = A_3 = \left\{ \frac{\text{Low}}{\text{Chevrolet}}, \frac{\text{Very Low}}{\text{Buick}}, \frac{\text{Very High}}{\text{Toyota}} \right\}$$

$$\text{repair frequency} = A_4 = \left\{ \frac{\text{Perfect}}{\text{Chevrolet}}, \frac{\text{Very High}}{\text{Buick}}, \frac{\text{Medium}}{\text{Toyota}} \right\}$$

Next we evaluate the importance of each objective:

$$\text{Importance of objectives} = \left\{ \frac{\text{Medium}}{A_1}, \frac{\text{High}}{A_2}, \frac{\text{Perfect}}{A_3}, \frac{\text{Medium}}{A_4} \right\}$$

Using this information we get

$$b_1 = \text{Medium} = S_3;\ b_2 = \text{High} = S_4;\ b_3 = \text{Perfect} = S_6;\ b_4 = \text{Medium} = S_3.$$

Since the negation in S is order reversing, $S_i' = S_{6-i}$ then

$$b_1' = S_3 = \text{Medium};\ b_2' = S_2 = \text{Low};\ b_3' = S_6 = \text{None};\ b_4' = \text{Medium} = S_3.$$

Since $C_i = b_i' \cup A_i$, then

$$C_1 = \text{Medium} \vee \left\{ \frac{\text{Medium}}{\text{Chevrolet}}, \frac{\text{Very High}}{\text{Buick}}, \frac{\text{Low}}{\text{Toyota}} \right\}$$
$$= \left\{ \frac{\text{Medium}}{\text{Chevrolet}}, \frac{\text{Very High}}{\text{Buick}}, \frac{\text{Medium}}{\text{Toyota}} \right\}$$
$$C_2 = \text{Low} \vee \left\{ \frac{\text{Medium}}{\text{Chevrolet}}, \frac{\text{Very Low}}{\text{Buick}}, \frac{\text{High}}{\text{Toyota}} \right\}$$
$$= \left\{ \frac{\text{Medium}}{\text{Chevrolet}}, \frac{\text{Very Low}}{\text{Buick}}, \frac{\text{High}}{\text{Toyota}} \right\}$$
$$C_3 = \text{None} \vee \left\{ \frac{\text{Low}}{\text{Chevrolet}}, \frac{\text{Very Low}}{\text{Buick}}, \frac{\text{Very High}}{\text{Toyota}} \right\}$$
$$= \left\{ \frac{\text{Low}}{\text{Chevrolet}}, \frac{\text{Very Low}}{\text{Buick}}, \frac{\text{Very High}}{\text{Toyota}} \right\}$$
$$C_4 = \text{Medium} \vee \left\{ \frac{\text{Perfect}}{\text{Chevrolet}}, \frac{\text{Very High}}{\text{Buick}}, \frac{\text{Medium}}{\text{Toyota}} \right\}$$
$$= \left\{ \frac{\text{Perfect}}{\text{Chevrolet}}, \frac{\text{Very High}}{\text{Buick}}, \frac{\text{Medium}}{\text{Toyota}} \right\}$$

Calculating $D = C_1 \cap C_2 \cap C_3 \cap C_4$, where $D(x) = \operatorname{Min}_i [C_i(x)]$ we get

$$D = \left\{ \frac{\text{Low}}{\text{Chevrolet}}, \frac{\text{Very Low}}{\text{Buick}}, \frac{\text{Medium}}{\text{Toyota}} \right\}.$$

Thus our selection based upon this model would be a Toyota.

The basic reason for selecting Toyota over Chevrolet, for example, is that Chevrolet gets a low rating in the most important criterion, gas mileage. If the Chevrolet had improved its rating in this criterion to medium, it would then be tied with the Toyota. Alternatively, if the criterion "gas mileage" had only been of, say, medium importance, then the effect of the low rating of Chevrolet in this criterion would not be as significant and the two would again tie. If, however, the criterion of comfort, in which the Toyota performed badly, were increased in importance, this would then also have the effect of tying the Toyota and Chevrolet.

A special procedure must be followed for selecting the optimal alternatives if we have ties. Assume that $x, y \in X$ are such that $D(x) = D(y) = \operatorname{Max}_{z \in X}(D(Z))$. Since $D(x) = \operatorname{Min}_i [C_i(x)]$ there exists some k such that $C_k(x) = D(x)$. Similarly there exists some g such that $C_g(y) = D(y)$. Let $\hat{D}(x) = \operatorname{Min}_{i \neq k} [C_i(x)]$, and let $\hat{D}(y) = \operatorname{Min}_{i \neq g} [C_i(y)]$. Then, we compare $\hat{D}(x)$ and $\hat{D}(y)$. If $\hat{D}(x) > \hat{D}(y)$, for example, we select x as our optimal. If, however, $\hat{D}(x) = \hat{D}(y)$, then there exists some r and e such that $\hat{D}(x) = C_r(x) = \hat{D}(y) = C_e(y)$. Then we formulate $\hat{\hat{D}}(x) = \operatorname{Min}_{i \neq k,r} [C_i(x)]$ and $\hat{\hat{D}}(y) = \operatorname{Min}_{i \neq g,e} [C_i(y)]$. We then compare $\hat{\hat{D}}(x)$ and $\hat{\hat{D}}(y)$. We continue in this manner. If after exhausting all the objectives we still cannot distinguish between x and y, then they are deemed tied.

There exists an alternative approach to adjudicating ties. Since the scale used to measure our preferences was not a very fine scale, we may allow the decision maker to use a refinement of the scale to help in ties.

Assume x and y are tied for the optimal value in D. Thus there exists some k such that $D(x) = C_k(x) = \operatorname{Min}_i [C_i(x)]$ and there exists some g such that $D(x) = C_g(y) = \operatorname{Min}_i C_i(y)$. We may now ask the decision maker to make a finer distinction between these two values. That is, though he selected the same $s \in S$ to evaluate $C_k(x)$ and $C_g(y)$, it may be possible for the decision maker to say that, for example, $C_k(x) > C_g(y)$. That is, it may be worth his while to expend the effort to make a finer distinction between these two situations now that he has reduced his decision to selection based upon this information.

DISCUSSION OF PROPERTIES OF THE MODEL

The proposed model works in the following manner. For a particular objective the negation of its importance acts as a barrier such that all ratings of alternatives that are below that barrier become equal to the value of that barrier. That is, we disregard all distinctions less than the barrier while keeping distinctions above this barrier. This works in the same manner as the classroom grading procedure of lumping all students whose grade averages fall below 60 into the F category while keeping distinctions of A, B, C, and D for students with grades of at least 60. In our model, however, this barrier varies, depending upon the importance of the objective. In particular, the more important the objective, the lower this barrier and thus the more levels of distinction. Hence, as an objective becomes less important we raise the distinction barrier, which "penalizes" the alternative less if it fails in this criterion. In the extreme, if the objective is totally unimportant, then the barrier is raised to its highest value and all alternatives are given the same rating and no distinction is made based on this criterion. If, however, the objective is most important, all distinctions are kept.