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Group Decision Making with Interval-Valued Intuitionistic Multiplicative Linguistic Preference Relations

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Abstract

To express the asymmetrically uncertain preferred and non-preferred qualitative judgments of decision makers, this paper introduces interval-valued intuitionistic multiplicative linguistic variables (IVIMLVs). To show their application in decision making, a ranking method is first offered. Then, we introduce IVIMLVs for preference relations and propose interval-valued intuitionistic multiplicative linguistic preference relations (IVIMLPRs). To obtain the ranking reasonably, a consistency definition for IVIMLPRs is presented. A mathematical optimization model for judging the consistency of IVIMLPRs based on the new concept is constructed. To address two general cases: incompleteness and inconsistency, mathematical optimization models for ascertaining unknown values in incomplete IVIMLPRs and deriving completely consistent IVIMLPRs from inconsistent ones are built, respectively. For group decision making, a consensus index is defined to measure the consensus achieved among the decision makers' preferences. If the consensus is not enough, a mathematical optimization model for improving the consensus level is established. Furthermore, a linear optimization model for determining the weights of the decision makers based on the consensus analysis is constructed. Finally, a group decision-making method with IVIMLPRs based on consistency and consensus analysis is offered, and its application on selecting supply chain cooperative partners is offered.

Keywords Group decision making · IVIMLPR · Consistency · Consensus · Mathematical optimization model

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1 Introduction

Decision making usually needs a group of decision makers (DMs) comparing and ranking objects. Preference relation is one of the most important decision-making methods. Researches about its theory and application have attracted significant attention. Because traditional preference relations require DMs to offer exact values (Saaty 1980; Tanino 1984), their application is restricted. To solve this issue, scholars introduced Zadeh's fuzzy sets for preference relations and developed decision making with fuzzy preference relations (Chiclana et al. 2009; Meng and Chen 2018; Saaty and Vargas 1987; van Laarhoven and Pedrycz 1983). Because intervals can simply denote the uncertainties of DMs, interval fuzzy preference relations are one of the most widely used preference relations. Following the construction of intervals, there are two types of interval fuzzy preference relations: additive interval fuzzy preference relations (An et al. 2018; Meng and Tan 2017; Wu et al. 2019a, b, c, d) and multiplicative interval fuzzy preference relations (Meng et al. 2017).

The same as other types of quantitative fuzzy variables, intervals are insufficient to address more complex situations. To better express the fuzziness of human subjective judgements, Zadeh (1975) introduced linguistic variables to denote the judgements of DMs, such as fast, slow, and fair. Later, researchers noted the advantages of linguistic variables and introduced them in preference relations, which is known as linguistic preference relations (Cabrerizo et al. 2017; Jin et al. 2016a, b; Xu 2004a). Considering the fact that linguistic variables cannot reflect the uncertainties of DMs, Xu (2004b) further introduced additive interval linguistic variables and additive interval linguistic preference relations (AILPRs). After that many decision-making methods with interval linguistic preference relations have been proposed. For example, Chen and Lee (2012) presented a group decision making (GDM) method with AILPRs based on the defined interval linguistic ordered weighted aggregation operator and the likelihood of individual AILPRs. Following the work of Xu (2005), Tapia García et al. (2012) proposed a GDM method with AILPRs based on the defined consensus measure and proximity measure. Using the 2-tuple linguistic representation model (Herrera and Martínez 2000), Xu and Wu (2013) introduced another GDM method with AILPRs based on consensus analysis and a model for determining the weights of DMs. Chen et al. (2011) introduced a compatible index for AILPRs and used this index to build a model for deriving the weights of DMs. Then, the authors proposed a new GDM method. A similar research can be seen in (Zhou and Chen 2013). Meng et al. (2016) analyzed the relationship between linguistic variables and interval linguistic variables. Then, the authors discussed the consistency of AILPRs following the consistency concepts of linguistic preference relations (Dong et al. 2008; Xu 2011). Subsequently, a consistency and consensus analysis-based algorithm for GDM with AILPRs is introduced. To address differences between operational laws on linguistic variables and interval linguistic variables, Meng et al. (2019a, b) presented the concepts of quasi interval linguistic variables and quasi interval linguistic preference relations (QILPRs). Then, the authors defined an additive consistency concept for AILPRs that satisfies all properties for the consistency concepts for linguistic preference relations (Dong et al. 2008; Xu 2011). Based on this concept, a GDM method with inconsistent and incomplete AILPRs is provided. On the other hand, Xu (2006) introduced multiplicative interval linguistic variables for preference relations and presented multiplicative interval linguistic preference relations (MILPRs). Then, the author offered us a GDM method with MILPRs based on the defined interval linguistic ordered weighted geometric mean operators. Zhou et al. (2014) presented an approach to GDM with MILPRs using the defined compatible index.

However, in some situations, more than one linguistic variable may exists for a judgement. To denote such cases, Zhu and Xu (2014) applied hesitant fuzzy linguistic term sets (HFLTSs) (Rodríguez et al. 2012) to define hesitant fuzzy linguistic preference relations (HFLPRs) and then discussed their consistency. Based on continuous linguistic term sets, Wang and Xu (2015) proposed extended HFLPRs. Then, the authors presented a method for ranking objects from extended HFLPRs based on the assumption that there is a uniform distribution on each hesitant fuzzy linguistic judgement. Furthermore, Wu and Xu (2016a) defined another additive consistency concept for HFLPRs based on the expectations of HFLTSs (Wu and Xu 2016b). Zhang and Wang (2014) noted that additive consistency concepts have some limitations and defined a multiplicative consistency concept for HFLPRs. Liu et al. (2019a, b, c, d) studied incomplete HFLPRs and presented a consistency improvement method. Different from HFLPRs, whose elements are defined on the symmetrical linguistic term sets, Tang and Meng (2019) presented the concept of multiplicative HFLTSs and introduced multiplicative hesitant fuzzy linguistic preference relations (MHFLPRs). Based on consistency and consensus analysis, the authors proposed a procedure for GDM with MHFLPRs. Tang et al. (2019) further researched decision making with multiplicative interval linguistic hesitant fuzzy preference relations (MILHFPRs) to denote the asymmetrically interval hesitant qualitative judgements. Note that all of the above mentioned fuzzy sets cannot denote the preferred and non-preferred judgements of DMs simultaneously. To cope with this issue, Atanassov's intuitionistic fuzzy sets (IFSs) (Atanassov 1986) are good choices that use a real value in [0, 1] to denote the membership and non-membership degrees of a judgement, respectively. Later, Atanassov and Gargov (1989) further introduced interval-valued intuitionistic fuzzy sets (IVIFSs) to express the uncertain membership and non-membership degrees of DMs. Szmidt and Kacprzyk (1988) noted the advantages of IFSs and presented intuitionistic fuzzy preference relations (IFPRs). More researches about decision making with IFPRs can be found in (Gong et al. 2009, 2011, 2018; Jin et al. 2019; Liu et al. 2019a, b, c, d; Xu 2007; Yang et al. 2019; Zhang and Pedrycz 2018). Furthermore, Meng et al. (2019a, b) studied decision making with intuitionistic linguistic preference relations that can denote the qualitative and quantitative intuitionistic preferences simultaneously. Jin et al. (2019) followed Liao and Xu's multiplicative consistency concept for IFPRs (Liao and Xu 2014) to offer a multiplicative consistency concept for intuitionistic linguistic preference relations and offered an iteration-based GDM method. Zhang and Pedrycz (2019) discussed the consistency of interval-valued intuitionistic multiplicative preference relations (IVIMPRs) and built several programming models to cope with the consistency and consensus.

However, when DMs can only offer their preferred and non-preferred qualitative judgments, all of the above introduced fuzzy sets are helpless. Considering this situation, this paper introduces the concept of interval-valued intuitionistic multiplicative linguistic variables (IVIMLVs) that use a multiplicative interval linguistic variable defined on the asymmetrical linguistic term sets to denote the uncertain preferred and non-preferred qualitative judgements of DMs, respectively. Then, interval-valued intuitionistic multiplicative linguistic preference relations (IVIMLPRs) are proposed. After that, this paper studies GDM with IVIMLPRs. The main highlights include: (i) a consistency concept, which satisfies upper triangular property and robustness, is defined; (ii) following this concept, mathematical optimization models for judging the consistency of IVIMLPRs are constructed; (iii) for incomplete and inconsistent IVIMLPRs, mathematical optimization models for ascertaining unknown values and deriving consistent IVIMLPRs are built, respectively; (iv) Based on quasi IVIMLPRs, a consensus index is proposed; (v) following the consensus analysis, a mathematical optimization model for determining the weights of DMs is established; (vi) when the consensus of individual judgements does not satisfy a minimum threshold, a mathematical optimization model for improving the consensus level is constructed; (vii) an algorithm for GDM with IVIMLPRs based on consistency and consensus is developed; (viii) a practical GDM problem on selecting partners in supply chain is offered to show the application of these new results.

The paper runs as follows: Sect. 2 contains the background results for our study, including multiplicative linguistic variables, multiplicative linguistic preference relations (MLPRs), multiplicative interval linguistic variables (MILVs), and multiplicative interval linguistic preference relations (MILPRs). Section 3 proposes the concept of IVIMLPRs and studies the consistency of IVIMLPRs. Section 4 discusses how to judge the consistency of IVIMLPRs using the built mathematical optimization model. Section 5 analyzes two usual cases: incomplete and inconsistent IVIMLPRs. To address these two types of IVIMLPRs, mathematical optimization models to obtain unknown linguistic variables and derive consistent IVIMLPRs are constructed, respectively. Section 6 focuses on GDM with IVIMLPRs and defines a consensus index. Then, two mathematical optimization model-based methods for determining the weights of DMs and improving the consensus level of individual IVIMLPRs are offered, respectively. Furthermore, an algorithm for GDM with incomplete and inconsistent IVIMLPRs is provided. Section 7 uses a practical example to show the application of the new algorithm. Conclusions are drawn in Sect. 8.

2 Background and Framework

For simplicity, let $X = \{x_1, x_2, \ldots, x_n\}$ denote the object set. To express asymmetrical qualitative judgements of DMs, Xu (2004a) introduced the concept of multiplicative linguistic variables (MLVs), which are defined on the discrete asymmetrical linguistic term set (DALTS) $S = \{s_a | a = 1/t, \ldots, 1/2, 1, 2, \ldots, t\}$. Any linguistic term s_λ expresses a value of MLVs with the following conditions:

(i) Ordered relationship: if $a_1 > a_2$, then $s_{a_1} > s_{a_2}$;
(ii) Reciprocity: for any $s_a \in S$, there is $s_b \in S$ such that $s_a \otimes s_b = s_{ab} = 1$.

For instance, a DALTS may be denoted as: $S = \{s_{1/3}$: very small, $s_{1/2}$: small, s_1: fair, s_2: big, s_3: very big$\}$. To avoid losing information in the process of calculation, Xu (2004a) extended the DALTS S to the continuous asymmetrical linguistic term set

(CALTS) $S' = \{s_b | b \in [1/t, t]\}$. For any $s_b \in S'$, it is called an original linguistic variable under the condition $s_b \in S$. Otherwise, it is a virtual linguistic variable. Generally speaking, virtual linguistic variables only appear in calculation. For any $s_b \in S'$, we let $I(s_b) = b$.

Let $s_a, s_b \in S'$, then (i) $s_a \otimes s_b = s_{ab}$; (ii) $(s_a)^\lambda = s_{a^\lambda}$ for any $\lambda \in [0, 1]$; (iii) $\log_\lambda(s_a) = s_{\log_\lambda(a)}$ (Xu 2004a). Based on MLVs, MLPRs are defined as follows:

Definition 1 (Xu 2004a) Let $R = (r_{ij})_{n\times n}$ be a linguistic fuzzy matrix on X for the DALTS S. R is called a MLPR if the followings are true:

$$r_{ij} \otimes r_{ji} = s_1 \quad r_{ii} = s_1,$$

for all $i, j = 1, 2, \ldots, n$, where $r_{ij} \in S$ denote the preferred qualitative degree of the object x_i over x_j.To derive the reasonable ranking of objects from MLPRs, Xu (2004a) introduced the following consistency concept for MLPRs:

Definition 2 (Xu 2004a). Let $R = (r_{ij})_{n\times n}$ be a MLPR on X for the DALTS S. R is consistent if

$$r_{ij} = r_{ik} \otimes r_{kj} \tag{1}$$

for all $i, k, j = 1, 2, \ldots, n$.To denote the uncertain qualitative judgements of DMs, Xu (2006) further introduced the concept of MILVs.

Definition 3 (Xu 2006). Let $\bar{s} = [s_a, s_b]$ such that $s_a, s_b \in S'$ and $s_a \le s_b$. Then, $\bar{s}$ is called a MILV.

Let $\bar{s}_1 = [s_{a_1}, s_{b_1}]$ and $\bar{s}_2 = [s_{a_2}, s_{b_2}]$ be any two MILVs, then several of their operations are listed as follows (Xu 2006):

(i) $\bar{s}_1 \otimes \bar{s}_2 = [s_{a_1 a_2}, s_{b_1 b_2}]$,
(ii) $\bar{s}_1^\lambda = [s_{a_1^\lambda}, s_{b_1^\lambda}]$, $\lambda \in [0, 1]$,
(iii) $\log_\lambda(\bar{s}_1) = [s_{\log_\lambda(a_1)}, s_{\log_\lambda(b_1)}]$, $\lambda \in [0, 1]$,
(iv) $(\bar{s}_1)^{-1} = [s_{1/b_1}, s_{1/a_1}]$.

Similar to MLPRs, Xu (2006) presented MILPRs as follows:

Definition 4 (Xu 2006). Let $\bar{R} = (\bar{r}_{ij})_{n\times n}$ be an interval linguistic fuzzy matrix on X for the CALTS S'. $\bar{R}$ is called a MILPR if

$$r_{ij}^L \otimes r_{ji}^U = r_{ij}^U \otimes r_{ji}^L = s_1, \quad r_{ii}^L = r_{ii}^U = s_1 \tag{2}$$

are true for all $i, j = 1, 2, \ldots, n$, where $\bar{r}_{ij} = [r_{ij}^L, r_{ij}^U]$ is the uncertain preferred qualitative degree of the object x_i over x_j such that $r_{ij}^L, r_{ij}^U \in S'$ and $r_{ij}^L \le r_{ij}^U$.

Considering differences between operational laws on MLVs and IMLVs, for example $\bar{s}_1 \otimes (\bar{s}_1)^{-1} \ne [s_1, s_1]$, we cannot define the consistency of MILPRs in a similar way to that of MLPRs. Following the work of Meng et al. (2017), we introduce the following consistency concept for quasi MILPRs (QMILPRs):

Definition 5 Let $\bar{R} = (\bar{r}_{ij})_{n\times n}$ be a MILPR on X for the CALTS S'. $\bar{Q} = (\bar{q}_{ij})_{n\times n}$ is called a QMILPR for $\bar{R}$ if

$$\begin{cases} \bar{q}_{ij} = [r_{ij}^L, r_{ij}^U] \\ \bar{q}_{ji} = [r_{ji}^U, r_{ji}^L] \end{cases} \vee \begin{cases} \bar{q}_{ij} = [r_{ij}^U, r_{ij}^L] \\ \bar{q}_{ji} = [r_{ji}^L, r_{ji}^U] \end{cases} \tag{3}$$

for all $i, j = 1, 2, \ldots, n$.

From the concept of QMILPRs, we easily derive $\bar{q}_{ij} \otimes \bar{q}_{ji} = [s_1, s_1]$ for per (i, j), namely, elements in QMILPRs satisfy reciprocity. This allows us to define the consistency of QMILPRs in a similar way to Definition 2 for MLPRs. On the other hand, elements of QMILPRs are obtained from that of associated IMLPRs. Thus, we can obtain the consistency of MILPRs following that of QMILPRs. Based on this point of view, the following consistency concept for MILPRs is presented:

Definition 6 Let $\bar{R} = (\bar{r}_{ij})_{n\times n}$ be a MILPR on X for the CALTS S'. It is consistent if there is an associated consistent QMILPR $\bar{Q} = (\bar{q}_{ij})_{n\times n}$, namely,

$$\bar{q}_{ij} = \bar{q}_{ik} \otimes \bar{q}_{kj} \tag{4}$$

for all $i, j = 1, 2, \ldots, n$.

Remark 1 Following the reciprocity of elements in QMILPRs, one can easily show that Definition 6 satisfies two important properties for consistency concepts: upper triangular property and robustness.

3 Interval-Valued Intuitionistic Multiplicative Linguistic Preference Relations

To express the asymmetrically uncertain preferred and non-preferred qualitative judgments of DMs simultaneously, this section introduces a new type of linguistic fuzzy sets: interval-valued intuitionistic multiplicative linguistic fuzzy sets (IVIMLFSs).

Definition 7 An IVIMLFS $\tilde{S}$ on X for the CALTS S' is defined as:

$$\tilde{S} = \left\{ \left\langle x_i, \left([s_{\mu^-}(x_i), s_{\mu^+}(x_i)], [s_{v^-}(x_i), s_{v^+}(x_i)] \right) \right\rangle, i = 1, 2, \ldots, n \right\} \tag{5}$$

where $[s_{\mu^-}(x_i),\ s_{\mu^+}(x_i)]$ and $[s_{v^-}(x_i),\ s_{v^+}(x_i)]$ are the uncertain preferred and non-preferred qualitative degrees of the object x_i over x_j such that $s_{\mu^-}(x_i) \otimes s_{v^+}(x_i) \le s_1$ and $s_{\mu^+}(x_i) \otimes s_{v^-}(x_i) \le s_1$. In addition, $\tilde{s} = \left([s_{\mu^-},\ s_{\mu^+}],\ [s_{v^-},\ s_{v^+}]\right)$ is an IVIMLV such that $s_{\mu^-} \otimes s_{v^+} \le s_1$ and $s_{\mu^+} \otimes s_{v^-} \le s_1$.

Now, we apply the following example to show the situations where IVIMLVs may be used and the advantages of IVIMLVs for representing the judgments of DMs.

A DM is invited to compare two brands of air conditioning. Because there are many factors, such as noise, energy consumption, appearance, price and brand effect, it is not an easy thing to offer his/her quantitative judgment. Linguistic variables are good choices to address this case. Let $S = \{s_{1/5}$: extremely bad, $s_{1/4}$: very bad, $s_{1/3}$: bad, $s_{1/2}$: a little bad, s_1: fair, s_2: a little good, s_3: good, s_4: very, s_5: extremely good$\}$ be the given DALTS. The DM can use linguistic variables in S to give his/her judgment. If the DM judges that the preferred degree of the first brand of air conditioning over the second one is between "$s_{1/2}$: a little bad" and "s_3: good", while it is between "$s_{1/4}$: a bit bad" and "s_1: fair" for the preferred degree of the second brand of air conditioning over the first one, namely, the non-preferred degree of the first brand of air conditioning over the second one is between "$s_{1/4}$: a bit bad" and "s_1: fair". To express these judgments, previous fuzzy variables are helpless, and IVIMLVs are good tools that can easily denote the above judgments, where $\tilde{s} = \big([s_{1/2}, s_3], [s_{1/4}, s_1]\big)$.

To derive the ranking of objects from IVIMLVs, the score and accuracy functions are defined as follows:

Definition 8 Let $\tilde{s} = \big([s_{\mu^-}, s_{\mu^+}], [s_{v^-}, s_{v^+}]\big)$ be an IVIMLV on X for the CALTS S'. Then, its score value is defined as:

$$V(\tilde{s}) = \frac{\mu^- \mu^+}{v^- v^+} \tag{6}$$

and the accuracy value is defined as:

$$A(\tilde{s}) = \big(\mu^- \mu^+\big)\big(v^- v^+\big) \tag{7}$$

Let $\tilde{s}_1 = \Big([s_{\mu_1^-}, s_{\mu_1^+}], [s_{v_1^-}, s_{v_1^+}]\Big)$ and $\tilde{s}_2 = \Big([s_{\mu_2^-}, s_{\mu_2^+}], [s_{v_2^-}, s_{v_2^+}]\Big)$ be any two IVIMLVs, then their order relationship is offered as follows:

(i) if $V(\tilde{s}_1) > V(\tilde{s}_2)$, then $\tilde{s}_1 > \tilde{s}_2$;

(ii) if $V(\tilde{s}_1) = V(\tilde{s}_2)$, then $\begin{cases} A(\tilde{s}_1) > A(\tilde{s}_2) \Rightarrow \tilde{s}_1 > \tilde{s}_2 \\ A(\tilde{s}_1) = A(\tilde{s}_2) \Rightarrow \tilde{s}_1 = \tilde{s}_2 \end{cases}$.

Now, we introduce the concept of IVIMLPRs whose elements are IVIMLVs.

Definition 9 An IVIMLPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ on X for the CALTS S' is defined as:

$$\begin{cases} s_{\mu_{ij}^-} = s_{v_{ji}^-}, s_{\mu_{ij}^+} = s_{v_{ji}^+} \\ s_{v_{ij}^-} = s_{\mu_{ji}^-}, s_{v_{ij}^+} = s_{\mu_{ji}^+} \\ s_{\mu_{ij}^-} \otimes s_{v_{ij}^+} \le s_1, s_{\mu_{ij}^+} \otimes s_{v_{ij}^-} \le s_1 \\ s_{\mu_{ii}^-} = s_{\mu_{ii}^+} = s_{v_{ii}^-} = s_{v_{ii}^+} = s_1 \end{cases} \tag{8}$$

for all $i, j = 1, 2, \ldots, n$, where $\tilde{r}_{ij} = \Big([s_{\mu_{ij}^-}, s_{\mu_{ij}^+}], [s_{v_{ij}^-}, s_{v_{ij}^+}]\Big)$ is an IVIMLV, and $[s_{\mu_{ij}^-}, s_{\mu_{ij}^+}]$ and $[s_{v_{ij}^-}, s_{v_{ij}^+}]$ are the uncertain preferred and non-preferred qualitative degrees of the object x_i over x_j, respectively.

For example, Let $X = \{x_1, x_2, x_3\}$ and $S' = \{s_b | 1/4 \le b \le 4\}$. Then, an IVIMLPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ on X for S' may be defined as:

$$\tilde{R} = \begin{pmatrix} ([s_1, s_1], [s_1, s_1]) & \big([s_{1/3}, s_1], [s_{1/2}, s_2]\big) & \big([s_2, s_4], [s_{1/4}, s_{1/3}]\big) \\ \big([s_{1/2}, s_2], [s_{1/3}, s_1]\big) & ([s_1, s_1], [s_1, s_1]) & \big([s_{1/4}, s_{1/2}], [s_1, s_3]\big) \\ \big([s_{1/4}, s_{1/3}], [s_2, s_4]\big) & \big([s_1, s_3], [s_{1/4}, s_{1/2}]\big) & ([s_1, s_1], [s_1, s_1]) \end{pmatrix}$$

Remark 2 When $\begin{cases} s_{\mu_{ij}^-} \otimes s_{v_{ij}^+} = s_1 \\ s_{\mu_{ij}^+} \otimes s_{v_{ij}^-} = s_1 \end{cases}$ for all $i, j = 1, 2, \ldots, n$, then the IVIMLPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ reduces to a MILPR $\bar{R} = (\bar{r}_{ij})_{n\times n}$[52].

To derive the reasonable ranking of objects from IVIMLPRs, the consistency analysis is indispensable. Therefore, we first introduce the concept of two-dimensional preferred multiplicative interval linguistic fuzzy variables (TDPMILFVs). Definition 9 shows that $[s_{v^-}, s_{v^+}]$ is the uncertain non-preferred qualitative degree for any $\tilde{s} = \big([s_{\mu^-}, s_{\mu^+}], [s_{v^-}, s_{v^+}]\big)$. Thus, s_{1/v^+} and s_{1/v^-} can be regarded as the lower and upper preferred degrees for $[s_{v^-}, s_{v^+}]$, respectively. For any IVIMLV $\tilde{s} = \big([s_{\mu^-}, s_{\mu^+}], [s_{v^-}, s_{v^+}]\big)$, its associated TDPMILFV is defined as:$\ddot{s} = \big([s_{\mu^-}, s_{1/v^+}], [s_{\mu^+}, s_{1/v^-}]\big)$. Following Definition 9, we have $s_{\mu_{ij}^-} \le s_{1/v_{ij}^+}$ and $s_{\mu_{ij}^+} \le s_{1/v_{ij}^-}$.

Definition 10 Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an IVIMLPR on X for the CALTS S'. $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ is called a two-dimensional preferred multiplicative interval linguistic preference relation (TDP-MILPR), where $\tilde{p}_{ij} = \Big([s_{\mu_{ij}^-}, s_{1/v_{ij}^+}], [s_{\mu_{ij}^+}, s_{1/v_{ij}^-}]\Big)$ is a TDP-MILFV for all $i, j = 1, 2, \ldots, n$ with the conditions as shown in Definition 9.

Following Definition 10, we further propose quasi TDPMILPRs (QTDP-MILPRs) in a similar way to QMILPRs.

Definition 11 Let $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ be a TDPMILPR for the IVIMLPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$, where $\tilde{p}_{ij} = \Big([s_{\mu_{ij}^-}, s_{1/v_{ij}^+}], [s_{\mu_{ij}^+}, s_{1/v_{ij}^-}]\Big)$ is a TDPMILFV for all $i, j = 1, 2, \ldots, n$. $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$ is called a QTDP-MILPR for $\tilde{P}$ if one of the following four cases is true:

$$\begin{aligned} &\begin{cases} \tilde{q}_{ij} = \Big([s_{\mu_{ij}^-}, s_{1/v_{ij}^+}], [s_{\mu_{ij}^+}, s_{1/v_{ij}^-}]\Big) \\ \tilde{q}_{ji} = \Big([s_{1/\mu_{ij}^-}, s_{v_{ij}^+}], [s_{1/\mu_{ij}^+}, s_{v_{ij}^-}]\Big) \end{cases}, \quad \begin{cases} \tilde{q}_{ij} = \Big([s_{1/v_{ij}^+}, s_{\mu_{ij}^-}], [s_{\mu_{ij}^+}, s_{1/v_{ij}^-}]\Big) \\ \tilde{q}_{ji} = \Big([s_{v_{ij}^+}, s_{1/\mu_{ij}^-}], [s_{1/\mu_{ij}^+}, s_{v_{ij}^-}]\Big) \end{cases} \\ &\begin{cases} \tilde{q}_{ij} = \Big([s_{\mu_{ij}^-}, s_{1/v_{ij}^+}], [s_{1/v_{ij}^-}, s_{\mu_{ij}^+}]\Big) \\ \tilde{q}_{ji} = \Big([s_{1/\mu_{ij}^-}, s_{v_{ij}^+}], [s_{v_{ij}^-}, s_{1/\mu_{ij}^+}]\Big) \end{cases}, \quad \begin{cases} \tilde{q}_{ij} = \Big([s_{1/v_{ij}^+}, s_{\mu_{ij}^-}], [s_{1/v_{ij}^-}, s_{\mu_{ij}^+}]\Big) \\ \tilde{q}_{ji} = \Big([s_{v_{ij}^+}, s_{1/\mu_{ij}^-}], [s_{v_{ij}^-}, s_{1/\mu_{ij}^+}]\Big) \end{cases} \end{aligned} \tag{9}$$

for per (i, j) such that $i < j$.

Definition 11 shows that a QTDP-MILPR $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ corresponds to two QMILPRs: $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ and $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$, where

$$\begin{cases} \bar{q}_{1,ij} = [s_{\mu_{ij}^-}, s_{1/v_{ij}^+}] \\ \bar{q}_{1,ji} = [s_{1/\mu_{ij}^-}, s_{v_{ij}^+}] \end{cases} \vee \begin{cases} \bar{q}_{1,ij} = [s_{1/v_{ij}^+}, s_{\mu_{ij}^-}] \\ \bar{q}_{1,ji} = [s_{v_{ij}^+}, s_{1/\mu_{ij}^-}] \end{cases} \tag{10}$$

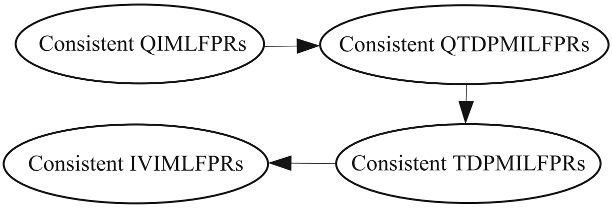

Fig. 1 The relationships of consistency

$$\begin{cases} \tilde{q}_{2,ij} = [s_{\mu_{ij}^{+}}, s_{1/v_{ij}^{-}}] \\ \tilde{q}_{2,ji} = [s_{1/\mu_{ij}^{+}}, s_{v_{ij}^{-}}] \end{cases} \vee \begin{cases} \tilde{q}_{2,ij} = [s_{1/v_{ij}^{-}}, s_{\mu_{ij}^{+}}] \\ \tilde{q}_{2,ji} = [s_{v_{ij}^{-}}, s_{1/\mu_{ij}^{+}}] \end{cases} \tag{11}$$

for per (i, j) such that $i < j$.

Similar to Definition 6, we define the consistency of QTDP-MILPRs using consistent QMILPRs.

Definition 12 Let $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ be a TDP-MILPR for the IVIMLPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$, and let $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$ be an associated QTDP-MILPR for $\tilde{P}$. If the QMILPRs $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ and $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$ for $\tilde{Q}$ as shown in formulae (10) and (11) are both consistent, then $\tilde{Q}$ is consistent.

Following the relationships between elements in IVIMLPRs, TDP-MILPR, and QTDP-MILPR, we present a consistency concept for IVIMLPRs as follow:

Definition 13 Let $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ be a TDP-MILPR for the IVIMLPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$.$\tilde{P}$ is consistent if there is a consistent QTDP-MILPR $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$ for $\tilde{P}$ based on Definition 12.

Definition 14 Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an IVIMLPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$. $\tilde{R}$ is consistent if its associated TDP-MILPR $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ is consistent based on Definition 13.

To understand the consistency relationships clearly, please see Fig. 1.

Remark 3 Following the properties of consistent QMILPRs, we derive that Definition 14 satisfies: *upper triangular property* and *robustness*.

4 Judging the Consistency of IVIMLPRs

Definition 14 indicates that we only need to judge the consistency of the QMILPRs $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ and $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$ as shown in formulae (10) and (11) for judging the consistency of $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$. Because there are many QMILPRs, we cannot directly use Definition 6 to derive the consistency of $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$. Considering the fact that we only need to judge whether a pair of associated consistent QMILPRs exists rather than judge the consistency of all associated QMILPRs, this section builds several optimization models to judge the consistency of IVIMLPRs based on the consistency of their QMILPRs.

For any given IVIMLPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$, if it is consistent following Definition 14, then we have

$$\begin{cases} \left([s_{\mu_{ij}^-},s_{1/v_{ij}^+}]\right)^{\alpha_{ij}}\otimes\left([s_{1/v_{ij}^+},s_{\mu_{ij}^-}]\right)^{1-\alpha_{ij}}=\left(\left([s_{\mu_{ik}^-},s_{1/v_{ik}^+}]\right)^{\alpha_{ik}}\otimes\left([s_{1/v_{ik}^+},s_{\mu_{ik}^-}]\right)^{1-\alpha_{ik}}\right)\otimes\left(\left([s_{\mu_{kj}^-},s_{1/v_{kj}^+}]\right)^{\alpha_{kj}}\otimes\left([s_{1/v_{kj}^+},s_{\mu_{kj}^-}]\right)^{1-\alpha_{kj}}\right) \\ \left([s_{\mu_{ij}^+},s_{1/v_{ij}^-}]\right)^{\beta_{ij}}\otimes\left([s_{1/v_{ij}^-},s_{\mu_{ij}^+}]\right)^{1-\beta_{ij}}=\left(\left([s_{\mu_{ik}^+},s_{1/v_{ik}^-}]\right)^{\beta_{ik}}\otimes\left([s_{1/v_{ik}^-},s_{\mu_{ik}^+}]\right)^{1-\beta_{ik}}\right)\otimes\left(\left([s_{\mu_{kj}^+},s_{1/v_{kj}^-}]\right)^{\beta_{kj}}\otimes\left([s_{1/v_{kj}^-},s_{\mu_{kj}^+}]\right)^{1-\beta_{kj}}\right) \end{cases} \tag{12}$$

for all i, k, $j=1, 2,\ldots, n$ such that $i<k<j$, where α_{ij} and β_{ij} are the 0–1 indicator variables for $\bar{q}_{1,ij}$ and $\bar{q}_{2,ij}$, respectively, denoted as

$$\alpha_{ij}=\begin{cases}1 & \bar{q}_{1,ij}=[s_{\mu_{ij}^-},\ s_{1/v_{ij}^+}] \\ 0 & \bar{q}_{1,ij}=[s_{1/v_{ij}^+},\ s_{\mu_{ij}^-}]\end{cases},\quad \beta_{ij}=\begin{cases}1 & \bar{q}_{2,ij}=[s_{\mu_{ij}^+},\ s_{1/v_{ij}^-}] \\ 0 & \bar{q}_{2,ij}=[s_{1/v_{ij}^-},\ s_{\mu_{ij}^+}]\end{cases}$$

Formula (12) shows that

$$\begin{cases} (s_{\mu_{ij}^-})^{\alpha_{ij}}\otimes(s_{1/v_{ij}^+})^{1-\alpha_{ij}}=\left((s_{\mu_{ik}^-})^{\alpha_{ik}}\otimes(s_{1/v_{ik}^+})^{1-\alpha_{ik}}\right)\otimes\left((s_{\mu_{kj}^-})^{\alpha_{kj}}\otimes(s_{1/v_{kj}^+})^{1-\alpha_{kj}}\right) \\ (s_{1/v_{ij}^+})^{\alpha_{ij}}\otimes(s_{\mu_{ij}^-})^{1-\alpha_{ij}}=\left((s_{1/v_{ik}^+})^{\alpha_{ik}}\otimes(s_{\mu_{ik}^-})^{1-\alpha_{ik}}\right)\otimes\left((s_{1/v_{kj}^+})^{\alpha_{kj}}\otimes(s_{\mu_{kj}^-})^{1-\alpha_{kj}}\right) \\ (s_{\mu_{ij}^+})^{\beta_{ij}}\otimes(s_{1/v_{ij}^-})^{1-\beta_{ij}}=\left((s_{\mu_{ik}^+})^{\beta_{ik}}\otimes(s_{1/v_{ik}^-})^{1-\beta_{ik}}\right)\otimes\left((s_{\mu_{kj}^+})^{\beta_{kj}}\otimes(s_{1/v_{kj}^-})^{1-\beta_{kj}}\right) \\ (s_{1/v_{ij}^-})^{\beta_{ij}}\otimes(s_{\mu_{ij}^+})^{1-\beta_{ij}}=\left((s_{1/v_{ik}^-})^{\beta_{ik}}\otimes(s_{\mu_{ik}^+})^{1-\beta_{ik}}\right)\otimes\left((s_{1/v_{kj}^-})^{\beta_{kj}}\otimes(s_{\mu_{kj}^+})^{1-\beta_{kj}}\right) \end{cases} \tag{13}$$

where i, k, $j=1, 2,\ldots, n$ such that $i<k<j$.

Formula (13) shows that

$$\begin{cases} I(s_{\mu_{ij}^-})^{\alpha_{ij}}\otimes I(s_{1/v_{ij}^+})^{1-\alpha_{ij}}=\left(I(s_{\mu_{ik}^-})^{\alpha_{ik}}\otimes I(s_{1/v_{ik}^+})^{1-\alpha_{ik}}\right)\otimes\left(I(s_{\mu_{kj}^-})^{\alpha_{kj}}\otimes I(s_{1/v_{kj}^+})^{1-\alpha_{kj}}\right) \\ I(s_{1/v_{ij}^+})^{\alpha_{ij}}\otimes I(s_{\mu_{ij}^-})^{1-\alpha_{ij}}=\left(I(s_{1/v_{ik}^+})^{\alpha_{ik}}\otimes I(s_{\mu_{ik}^-})^{1-\alpha_{ik}}\right)\otimes\left(I(s_{1/v_{kj}^+})^{\alpha_{kj}}\otimes I(s_{\mu_{kj}^-})^{1-\alpha_{kj}}\right) \\ I(s_{\mu_{ij}^+})^{\beta_{ij}}\otimes I(s_{1/v_{ij}^-})^{1-\beta_{ij}}=\left(I(s_{\mu_{ik}^+})^{\beta_{ik}}\otimes I(s_{1/v_{ik}^-})^{1-\beta_{ik}}\right)\otimes\left(I(s_{\mu_{kj}^+})^{\beta_{kj}}\otimes I(s_{1/v_{kj}^-})^{1-\beta_{kj}}\right) \\ I(s_{1/v_{ij}^-})^{\beta_{ij}}\otimes I(s_{\mu_{ij}^+})^{1-\beta_{ij}}=\left(I(s_{1/v_{ik}^-})^{\beta_{ik}}\otimes I(s_{\mu_{ik}^+})^{1-\beta_{ik}}\right)\otimes\left(I(s_{1/v_{kj}^-})^{\beta_{kj}}\otimes I(s_{\mu_{kj}^+})^{1-\beta_{kj}}\right) \end{cases} \tag{14}$$

namely,

$$\begin{cases} (\mu_{ij}^-)^{\alpha_{ij}}\times(1/v_{ij}^+)^{1-\alpha_{ij}}=\left((\mu_{ik}^-)^{\alpha_{ik}}\times(1/v_{ik}^+)^{1-\alpha_{ik}}\right)\left((\mu_{kj}^-)^{\alpha_{kj}}\times(1/v_{kj}^+)^{1-\alpha_{kj}}\right) \\ (1/v_{ij}^+)^{\alpha_{ij}}\times(\mu_{ij}^-)^{1-\alpha_{ij}}=\left((1/v_{ik}^+)^{\alpha_{ik}}\times(\mu_{ik}^-)^{1-\alpha_{ik}}\right)\left((1/v_{kj}^+)^{\alpha_{kj}}\times(\mu_{kj}^-)^{1-\alpha_{kj}}\right) \\ (\mu_{ij}^+)^{\beta_{ij}}\times(1/v_{ij}^-)^{1-\beta_{ij}}=\left((\mu_{ik}^+)^{\beta_{ik}}\times(1/v_{ik}^-)^{1-\beta_{ik}}\right)\left((\mu_{kj}^+)^{\beta_{kj}}\times(1/v_{kj}^-)^{1-\beta_{kj}}\right) \\ (1/v_{ij}^-)^{\beta_{ij}}\times(\mu_{ij}^+)^{1-\beta_{ij}}=\left((1/v_{ik}^-)^{\beta_{ik}}\times(\mu_{ik}^+)^{1-\beta_{ik}}\right)\left((1/v_{kj}^-)^{\beta_{kj}}\times(\mu_{kj}^+)^{1-\beta_{kj}}\right) \end{cases} \tag{15}$$

where i, k, $j=1, 2,\ldots, n$ such that $i<k<j$.

Taking the logarithm for each formula (15), we derive

$$\begin{cases} \alpha_{ij}\log(\mu_{ij}^-)+(1-\alpha_{ij})\log(1/v_{ij}^+)=\left(\alpha_{ik}\log(\mu_{ik}^-)+(1-\alpha_{ik})\log(1/v_{ik}^+)\right)+\left(\alpha_{kj}\log(\mu_{kj}^-)+(1-\alpha_{kj})\log(1/v_{kj}^+)\right)\\ \alpha_{ij}\log(1/v_{ij}^+)+(1-\alpha_{ij})\log(\mu_{ij}^-)=\left(\alpha_{ik}\log(1/v_{ik}^+)+(1-\alpha_{ik})\log(\mu_{ik}^-)\right)+\left(\alpha_{kj}\log(1/v_{kj}^+)+(1-\alpha_{kj})\log(\mu_{kj}^-)\right)\\ \beta_{ij}\log(\mu_{ij}^+)+(1-\beta_{ij})\log(1/v_{ij}^-)=\left(\beta_{ik}\log(\mu_{ik}^+)+(1-\beta_{ik})\log(1/v_{ik}^-)\right)+\left(\beta_{kj}\log(\mu_{kj}^+)+(1-\beta_{kj})\log(1/v_{kj}^-)\right)\\ \beta_{ij}\log(1/v_{ij}^-)+(1-\beta_{ij})\log(\mu_{ij}^+)=\left(\beta_{ik}\log(1/v_{ik}^-)+(1-\beta_{ik})\log(\mu_{ik}^+)\right)+\left(\beta_{kj}\log(1/v_{kj}^-)+(1-\beta_{kj})\log(\mu_{kj}^+)\right) \end{cases} \tag{16}$$

namely,

$$\begin{cases} \alpha_{ij}\left(\log(\mu_{ij}^-)+\log(v_{ij}^+)\right)-\log(v_{ij}^+)=\left(\alpha_{ik}\left(\log(\mu_{ik}^-)+\log(v_{ik}^+)\right)-\log(v_{ik}^+)\right)+\left(\alpha_{kj}\left(\log(\mu_{kj}^-)+\log(v_{kj}^+)\right)-\log(v_{kj}^+)\right)\\ \log(\mu_{ij}^-)-\alpha_{ij}\left(\log(v_{ij}^+)+\log(\mu_{ij}^-)\right)=\left(\log(\mu_{ik}^-)-\alpha_{ik}\left(\log(v_{ik}^+)+\log(\mu_{ik}^-)\right)\right)+\left(\log(\mu_{kj}^-)-\alpha_{kj}\left(\log(v_{kj}^+)+\log(\mu_{kj}^-)\right)\right)\\ \beta_{ij}\left(\log(\mu_{ij}^+)+\log(v_{ij}^-)\right)-\log(v_{ij}^-)=\left(\beta_{ik}\left(\log(\mu_{ik}^+)+\log(v_{ik}^-)\right)-\log(v_{ik}^-)\right)+\left(\beta_{kj}\left(\log(\mu_{kj}^+)+\log(v_{kj}^-)\right)-\log(v_{kj}^-)\right)\\ \log(\mu_{ij}^+)-\left(\beta_{ij}\log(v_{ij}^-)+\log(\mu_{ij}^+)\right)=\left(\log(\mu_{ik}^+)-\beta_{ik}\left(\log(v_{ik}^-)+\log(\mu_{ik}^+)\right)\right)+\left(\log(\mu_{kj}^+)-\beta_{kj}\left(\log(v_{kj}^-)+\log(\mu_{kj}^+)\right)\right) \end{cases} \tag{17}$$

Based on formula (17), we construct the following optimization model to judge the consistency of any given IVIMLPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$:

$$f^*=\min\sum_{i=1}^{n-2}\sum_{k=i+1}^{n-1}\sum_{j=k+1}^{n}\left(\varepsilon_{k,ij}^++\varepsilon_{k,ij}^-+\delta_{k,ij}^++\delta_{k,ij}^-+\theta_{k,ij}^++\theta_{k,ij}^-+\vartheta_{k,ij}^++\vartheta_{k,ij}^-\right)$$

$$s.t.\begin{cases} \left(\alpha_{ik}\left(\log(\mu_{ik}^-)+\log(v_{ik}^+)\right)-\log(v_{ik}^+)\right)+\left(\alpha_{kj}\left(\log(\mu_{kj}^-)+\log(v_{kj}^+)\right)\right.\\ \quad\left.-\log(v_{kj}^+)\right)=\alpha_{ij}\left(\log(\mu_{ij}^-)+\log(v_{ij}^+)\right)-\log(v_{ij}^+)-\varepsilon_{k,ij}^++\varepsilon_{k,ij}^-\\ \left(\log(\mu_{ik}^-)-\alpha_{ik}\left(\log(v_{ik}^+)+\log(\mu_{ik}^-)\right)\right)+\left(\log(\mu_{kj}^-)-\alpha_{kj}\left(\log(v_{kj}^+)\right.\right.\\ \left.\left.+\log(\mu_{kj}^-)\right)\right)=\log(\mu_{ij}^-)-\alpha_{ij}\left(\log(v_{ij}^+)+\log(\mu_{ij}^-)\right)-\delta_{k,ij}^++\delta_{k,ij}^-\\ \left(\beta_{ik}\left(\log(\mu_{ik}^+)+\log(v_{ik}^-)\right)-\log(v_{ik}^-)\right)+\left(\beta_{kj}\left(\log(\mu_{kj}^+)+\log(v_{kj}^-)\right)\right.\\ \quad\left.-\log(v_{kj}^-)\right)=\beta_{ij}\left(\log(\mu_{ij}^+)+\log(v_{ij}^-)\right)-\log(v_{ij}^-)-\theta_{k,ij}^++\theta_{k,ij}^-\\ \left(\log(\mu_{ik}^+)-\beta_{ik}\left(\log(v_{ik}^-)+\log(\mu_{ik}^+)\right)\right)+\left(\log(\mu_{kj}^+)-\beta_{kj}\left(\log(v_{kj}^-)\right.\right.\\ \left.\left.+\log(\mu_{kj}^+)\right)\right)=\log(\mu_{ij}^+)-\left(\beta_{ij}\log(v_{ij}^-)+\log(\mu_{ij}^+)\right)-\vartheta_{k,ij}^++\vartheta_{k,ij}^-\\ \alpha_{ij}=0\vee 1,\beta_{ij}=0\vee 1,i,j=1,2,\ldots,n,i<j\\ \varepsilon_{k,ij}^+,\varepsilon_{k,ij}^-,\delta_{k,ij}^+,\delta_{k,ij}^-,\theta_{k,ij}^+,\theta_{k,ij}^-,\vartheta_{k,ij}^+,\vartheta_{k,ij}^-\geq 0,i,k,j=1,2,\ldots,n,\\ i<k<j \end{cases} \tag{M-1}$$

By solving model (M-1), if $f^*=0$, then formula (17) is true, namely, formula (12) is true. Thus, $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ is consistent. Otherwise, it is inconsistent.

5 Incomplete and Inconsistent IVIMLPRs

In decision making with preference relations, two cases are usually encountered: incomplete and inconsistent preference relations (Capuano et al. 2018; Ureña et al. 2015a, b). To extend the application of IVIMLPRs in decision making, this section discusses incomplete and inconsistent IVIMLPRs.

5.1 Incomplete IVIMLPRs

In the procedure of decision making, some judging information may be unknown due to various reasons (Liu et al. 2019a, b, c, d; Sahu and Gupta 2018; Tang and Meng 2018; Wu et al. 2019a; Zhang et al. 2018a, b). To derive ranking of objects, it is necessary to determine unknown information. Thus, this subsection establishes an optimization model to determine unknown values based on the consistency analysis and the known information.

Property 1 *Let $\bar{Q} = (\bar{q}_{ij})_{n\times n}$ be a QMILPR for the MILPR $\bar{R} = (\bar{r}_{ij})_{n\times n}$. Then, it is consistent if and only if*

$$\bar{q}_{ij} = \sqrt[n]{\otimes_{k=1}^{n}\left(\bar{q}_{ik} \otimes \bar{q}_{kj}\right)} \tag{18}$$

for all $i, j = 1, 2,\ldots, n$ such that $i < j$.

Proof If $\bar{Q} = (\bar{q}_{ij})_{n\times n}$ is consistent, then formula (18) holds following formula (4). When formula (18) holds, we have

$$\bar{q}_{ij} = \sqrt[n]{\otimes_{k=1}^{n}\left(\bar{q}_{ik} \otimes \bar{q}_{kj}\right)} = \sqrt[n]{\otimes_{l=1}^{n}\left(\bar{q}_{il} \otimes \bar{q}_{lk} \otimes \bar{q}_{kl} \otimes \bar{q}_{lj}\right)}$$

$$= \sqrt[n]{\otimes_{l=1}^{n}\left(\bar{q}_{il} \otimes \bar{q}_{lk}\right)} \otimes \sqrt[n]{\otimes_{l=1}^{n}\left(\bar{q}_{kl} \otimes \bar{q}_{lj}\right)} = \bar{q}_{ik} \otimes \bar{q}_{kj}$$

following $\bar{q}_{ij} \otimes \bar{q}_{ji} = [s_1, s_1]$ for all $i, j = 1, 2, \ldots, n$. Thus, $\bar{Q} = (\bar{q}_{ij})_{n\times n}$ is consistent. □

Following Property 1, we discuss how to build an optimization model for determining unknown information. For the given incomplete IVIMLPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$, if there are linguistic variables in the CALTS S' that make the incomplete IVIMLPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be consistent, then two associated incomplete QMILPR $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ and $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$ are consistent. Following formula (18), we have

$$\begin{cases} \left(\bar{q}_{1,ij}\right)^{n-2} = \otimes_{k=1,k\neq i,j}^{n}\left(\bar{q}_{1,ik} \otimes \bar{q}_{1,kj}\right) \\ \left(\bar{q}_{2,ij}\right)^{n-2} = \otimes_{k=1,k\neq i,j}^{n}\left(\bar{q}_{2,ik} \otimes \bar{q}_{2,kj}\right) \end{cases} \tag{19}$$

for all $i, j = 1, 2, \ldots, n$ such that $i < j$.

Formula (19) shows that

$$\begin{cases} \left(\left([s_{\mu_{ij}^-}, s_{1/v_{ij}^+}]\right)^{\alpha_{ij}} \otimes \left([s_{1/v_{ij}^+}, s_{\mu_{ij}^-}]\right)^{1-\alpha_{ij}}\right)^{n-2} = \otimes_{k=1,k\neq i,j}^{n}\left(\left(\left([s_{\mu_{ik}^-}, s_{1/v_{ik}^+}]\right)^{\alpha_{ik}}\right.\right. \\ \left.\otimes\left([s_{1/v_{ik}^+}, s_{\mu_{ik}^-}]\right)^{1-\alpha_{ik}}\right) \otimes \left(\left([s_{\mu_{kj}^-}, s_{1/v_{kj}^+}]\right)^{\alpha_{kj}} \otimes \left([s_{1/v_{kj}^+}, s_{\mu_{kj}^-}]\right)^{1-\alpha_{kj}}\right)\Big) \\ \left(\left([s_{\mu_{ij}^+}, s_{1/v_{ij}^-}]\right)^{\beta_{ij}} \otimes \left([s_{1/v_{ij}^-}, s_{\mu_{ij}^+}]\right)^{1-\beta_{ij}}\right)^{n-2} = \otimes_{k=1,k\neq i,j}^{n}\left(\left(\left([s_{\mu_{ik}^+}, s_{1/v_{ik}^-}]\right)^{\beta_{ik}}\right.\right. \\ \left.\otimes\left([s_{1/v_{ik}^-}, s_{\mu_{ik}^+}]\right)^{1-\beta_{ik}}\right) \otimes \left(\left([s_{\mu_{kj}^+}, s_{1/v_{kj}^-}]\right)^{\beta_{kj}} \otimes \left([s_{1/v_{kj}^-}, s_{\mu_{kj}^+}]\right)^{1-\beta_{kj}}\right)\Big) \end{cases} \tag{20}$$

for all $i, j = 1, 2, \ldots, n$ such that $i < j$.

For per (i, j), following formula (20) we derive

$$\begin{cases} \otimes_{k=1,k\neq i,j}^{n}\left(\left(\left([I(s_{\mu_{ik}^-}), I(s_{1/v_{ik}^+})]\right)^{\alpha_{ik}} \otimes \left([I(s_{1/v_{ik}^+}), I(s_{\mu_{ik}^-})]\right)^{1-\alpha_{ik}}\right)\otimes\right. \\ \left.\left(\left([I(s_{\mu_{kj}^-}), I(s_{1/v_{kj}^+})]\right)^{\alpha_{kj}} \otimes \left([I(s_{1/v_{kj}^+}), I(s_{\mu_{kj}^-})]\right)^{1-\alpha_{kj}}\right)\right) \\ = \left(\left([I(s_{\mu_{ij}^-}), I(s_{1/v_{ij}^+})]\right)^{\alpha_{ij}} \otimes \left([I(s_{1/v_{ij}^+}), I(s_{\mu_{ij}^-})]\right)^{1-\alpha_{ij}}\right)^{n-2} \\ \otimes_{k=1,k\neq i,j}^{n}\left(\left(\left([I(s_{\mu_{ik}^+}), I(s_{1/v_{ik}^-})]\right)^{\beta_{ik}} \otimes \left([I(s_{1/v_{ik}^-}), I(s_{\mu_{ik}^+})]\right)^{1-\beta_{ik}}\right)\right. \\ \left.\otimes\left(\left([I(s_{\mu_{kj}^+}), I(s_{1/v_{kj}^-})]\right)^{\beta_{kj}} \otimes \left([I(s_{1/v_{kj}^-}), I(s_{\mu_{kj}^+})]\right)^{1-\beta_{kj}}\right)\right) \\ = \left(\left([I(s_{\mu_{ij}^+}), I(s_{1/v_{ij}^-})]\right)^{\beta_{ij}} \otimes \left([I(s_{1/v_{ij}^-}), I(s_{\mu_{ij}^+})]\right)^{1-\beta_{ij}}\right)^{n-2} \end{cases} \tag{21}$$

namely,

$$\begin{cases} \Pi_{k=1,k\neq i,j}^{n}\Big((\mu_{ik}^{-})^{\alpha_{ik}} \times (1/v_{ik}^{+})^{1-\alpha_{ik}} \times (\mu_{kj}^{-})^{\alpha_{kj}} \times (1/v_{kj}^{+})^{1-\alpha_{kj}}\Big) = \Big((\mu_{ij}^{-})^{\alpha_{ij}} \times (1/v_{ij}^{+})^{1-\alpha_{ij}}\Big)^{n-2} \\ \Pi_{k=1,k\neq i,j}^{n}\Big((1/v_{ik}^{+})^{\alpha_{ik}} \times (\mu_{ik}^{-})^{1-\alpha_{ik}} \times (1/v_{kj}^{+})^{\alpha_{kj}} \times (\mu_{kj}^{-})^{1-\alpha_{kj}}\Big) = \Big((1/v_{ij}^{+})^{\alpha_{ij}} \times (\mu_{ij}^{-})^{1-\alpha_{ij}}\Big)^{n-2} \\ \Pi_{k=1,k\neq i,j}^{n}\Big((\mu_{ik}^{+})^{\beta_{ik}} \times (1/v_{ik}^{-})^{1-\beta_{ik}} \times (\mu_{kj}^{+})^{\beta_{kj}} \times (1/v_{kj}^{-})^{1-\beta_{kj}}\Big) = \Big((\mu_{ij}^{+})^{\beta_{ij}} \times (1/v_{ij}^{-})^{1-\beta_{ij}}\Big)^{n-2} \\ \Pi_{k=1,k\neq i,j}^{n}\Big((1/v_{ik}^{-})^{\beta_{ik}} \times (\mu_{ik}^{+})^{1-\beta_{ik}} \times (1/v_{kj}^{-})^{\beta_{kj}} \times (\mu_{kj}^{+})^{1-\beta_{kj}}\Big) = \Big((1/v_{ij}^{-})^{\beta_{ij}} \times (\mu_{ij}^{+})^{1-\beta_{ij}}\Big)^{n-2} \end{cases} \tag{22}$$

-

Taking the logarithm for formula (24), we obtain

$$\begin{cases} \sum_{k=1,k\neq i,j}^{n}\Big(\alpha_{ik}\big(\log(\mu_{ik}^{-})+\log(v_{ik}^{+})\big)-\log(v_{ik}^{+})-\log(v_{kj}^{+})+ \\ \alpha_{kj}\Big(\log(\mu_{kj}^{-})+\log(v_{kj}^{+})\Big)\Big) = (n-2)\Big(\alpha_{ij}\Big(\log(\mu_{ij}^{-})+\log(v_{ij}^{+})\Big)-\log(v_{ij}^{+})\Big) \\ \sum_{k=1,k\neq i,j}^{n}\Big(\log(\mu_{ik}^{-})-\alpha_{ik}\big(\log(\mu_{ik}^{-})+\log(v_{ik}^{+})\big)+\log(\mu_{kj}^{-})- \\ \alpha_{kj}\Big(\log(\mu_{kj}^{-})+\log(v_{kj}^{+})\Big)\Big) = (n-2)\Big(\log(\mu_{ij}^{-})-\alpha_{ij}\Big(\log(\mu_{ij}^{-})+\log(v_{ij}^{+})\Big)\Big) \\ \sum_{k=1,k\neq i,j}^{n}\Big(\beta_{ik}\big(\log(\mu_{ik}^{+})+\log(v_{ik}^{-})\big)-\log(v_{ik}^{-})-\log(v_{kj}^{-})+ \\ \beta_{kj}\Big(\log(\mu_{kj}^{+})+\log(v_{kj}^{-})\Big)\Big) = (n-2)\Big(\beta_{ij}\Big(\log(\mu_{ij}^{+})+\log(v_{ij}^{-})\Big)-\log(v_{ij}^{-})\Big) \\ \sum_{k=1,k\neq i,j}^{n}\Big(\log(\mu_{ik}^{+})-\beta_{ik}\big(\log(\mu_{ik}^{+})+\log(v_{ik}^{-})\big)+\log(\mu_{kj}^{+})- \\ \beta_{kj}\Big(\log(\mu_{kj}^{+})+\log(v_{kj}^{-})\Big)\Big) = (n-2)\Big(\log(\mu_{ij}^{+})-\beta_{ij}\Big(\log(\mu_{ij}^{+})+\log(v_{ij}^{-})\Big)\Big) \end{cases} \tag{23}$$

Following the construction of elements in IVIMLPRs, we can only apply elements in the upper triangular parts of incomplete QMILPR $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ and $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$ to express each incomplete unknown judgment. Taking the first equation in formula (23) for example, we get

$$\begin{aligned} &\sum_{k=1,k\neq i,j}^{n}\Big(\alpha_{ik}\left(\log(\mu_{ik}^{-})+\log(v_{ik}^{+})\right)-\log(v_{ik}^{+})-\log(v_{kj}^{+})+\ \alpha_{kj}\Big(\log(\mu_{kj}^{-})+\log(v_{kj}^{+})\Big)\Big) \\ &\quad = \Big(\sum_{k=1}^{i-1}+\sum_{k=i+1}^{j}+\sum_{k=j+1}^{n}\Big) \\ &\qquad \times\Big(\alpha_{ik}\left(\log(\mu_{ik}^{-})+\log(v_{ik}^{+})\right)-\log(v_{ik}^{+})+\alpha_{kj}\Big(\log(\mu_{kj}^{-})+\log(v_{kj}^{+})\Big)\ -\log(v_{kj}^{+})\Big) \\ &\quad = \sum_{k=1}^{i-1}\Big(\log(v_{ki}^{+})-\alpha_{ki}\left(\log(\mu_{ki}^{-})+\log(v_{ki}^{+})\right)-\log(v_{kj}^{+}) \\ &\qquad +\alpha_{kj}\Big(\log(\mu_{kj}^{-})+\log(v_{kj}^{+})\Big)\Big) \\ &\qquad +\sum_{k=i+1}^{j}\left(\alpha_{ik}\left(\log(\mu_{ik}^{-})+\log(v_{ik}^{+})\right)-\ \log(v_{ik}^{+})+\alpha_{kj}\Big(\log(\mu_{kj}^{-})+\log(v_{kj}^{+})\Big)-\log(v_{kj}^{+})\right) \\ &\qquad +\sum_{k=j+1}^{n}\left(\alpha_{ik}\quad\times\left(\log(\mu_{ik}^{-})\ +\log(v_{ik}^{+})\right)-\log(v_{ik}^{+})+\log(v_{jk}^{+})-\alpha_{jk}\Big(\log(\mu_{jk}^{-})+\log(v_{jk}^{+})\Big)\right. \end{aligned}$$

Thus, formula (23) can be equivalently transformed into the following:

$$\begin{cases} \sum_{k=1}^{i-1} \varsigma_{i,k,j}^{1} + \sum_{k=i+1}^{j} \varsigma_{i,k,j}^{2} + \sum_{k=j+1}^{n} \varsigma_{i,k,j}^{3} = (n-2)\Big(\alpha_{ij}\Big(\log(\mu_{ij}^{-}) + \log(v_{ij}^{+})\Big) - \log(v_{ij}^{+})\Big) \\ \sum_{k=1}^{i-1} \varsigma_{i,k,j}^{4} + \sum_{k=i+1}^{j} \varsigma_{i,k,j}^{5} + \sum_{k=j+1}^{n} \varsigma_{i,k,j}^{6} = (n-2)\Big(\log(\mu_{ij}^{-}) - \alpha_{ij}\Big(\log(\mu_{ij}^{-}) + \log(v_{ij}^{+})\Big)\Big) \\ \sum_{k=1}^{i-1} \upsilon_{i,k,j}^{1} + \sum_{k=i+1}^{j} \upsilon_{i,k,j}^{2} + \sum_{k=j+1}^{n} \upsilon_{i,k,j}^{3} = (n-2)\Big(\beta_{ij}\Big(\log(\mu_{ij}^{+}) + \log(v_{ij}^{-})\Big) - \log(v_{ij}^{-})\Big) \\ \sum_{k=1}^{i-1} \upsilon_{i,k,j}^{4} + \sum_{k=i+1}^{j} \upsilon_{i,k,j}^{5} + \sum_{k=j+1}^{n} \upsilon_{i,k,j}^{6} = (n-2)\Big(\log(\mu_{ij}^{+}) - \beta_{ij}\Big(\log(\mu_{ij}^{+}) + \log(v_{ij}^{-})\Big)\Big) \end{cases} \tag{24}$$

where

$$\begin{aligned}
\varsigma_{i,k,j}^{1} &= \log(v_{ki}^{+}) - \alpha_{ki}\big(\log(\mu_{ki}^{-}) + \log(v_{ki}^{+})\big) + \alpha_{kj}\Big(\log(\mu_{kj}^{-}) + \log(v_{kj}^{+})\Big) - \log(v_{kj}^{+}), \\
\varsigma_{i,k,j}^{2} &= \alpha_{ik}\big(\log(\mu_{ik}^{-}) + \log(v_{ik}^{+})\big) - \log(v_{ik}^{+}) + \alpha_{kj}\Big(\log(\mu_{kj}^{-}) + \log(v_{kj}^{+})\Big) - \log(v_{kj}^{+}), \\
\varsigma_{i,k,j}^{3} &= \alpha_{ik}\big(\log(\mu_{ik}^{-}) + \log(v_{ik}^{+})\big) - \log(v_{ik}^{+}) + \log(v_{jk}^{+}) - \alpha_{jk}\Big(\log(\mu_{jk}^{-}) + \log(v_{jk}^{+})\Big), \\
\varsigma_{i,k,j}^{4} &= \alpha_{ki}\big(\log(\mu_{ki}^{-}) + \log(v_{ki}^{+})\big) - \log(\mu_{ki}^{-}) + \log(\mu_{kj}^{-}) - \alpha_{kj}\Big(\log(\mu_{kj}^{-}) + \log(v_{kj}^{+})\Big), \\
\varsigma_{i,k,j}^{5} &= \log(\mu_{ik}^{-}) - \alpha_{ik}\big(\log(\mu_{ik}^{-}) + \log(v_{ik}^{+})\big) + \log(\mu_{kj}^{-}) - \alpha_{kj}\Big(\log(\mu_{kj}^{-}) + \log(v_{kj}^{+})\Big), \\
\varsigma_{i,k,j}^{6} &= \log(\mu_{ik}^{-}) - \alpha_{ik}\big(\log(\mu_{ik}^{-}) + \log(v_{ik}^{+})\big) + \alpha_{jk}\Big(\log(\mu_{jk}^{-}) + \log(v_{jk}^{+})\Big) - \log(\mu_{jk}^{-}), \\
\upsilon_{i,k,j}^{1} &= \log(v_{ki}^{-}) - \beta_{ki}\big(\log(\mu_{ki}^{+}) + \log(v_{ki}^{-})\big) + \beta_{kj}\Big(\log(\mu_{kj}^{+}) + \log(v_{kj}^{-})\Big) - \log(v_{kj}^{-}), \\
\upsilon_{i,k,j}^{2} &= \beta_{ik}\big(\log(\mu_{ik}^{+}) + \log(v_{ik}^{-})\big) - \log(v_{ik}^{-}) + \beta_{kj}\Big(\log(\mu_{kj}^{+}) + \log(v_{kj}^{-})\Big) - \log(v_{kj}^{-}), \\
\upsilon_{i,k,j}^{3} &= \beta_{ik}\big(\log(\mu_{ik}^{+}) + \log(v_{ik}^{-})\big) - \log(v_{ik}^{-}) + \log(v_{jk}^{-}) - \beta_{jk}\Big(\log(\mu_{jk}^{+}) + \log(v_{jk}^{-})\Big), \\
\upsilon_{i,k,j}^{4} &= \beta_{ki}\big(\log(\mu_{ki}^{+}) + \log(v_{ki}^{-})\big) - \log(\mu_{ki}^{+}) + \log(\mu_{kj}^{+}) - \beta_{kj}\Big(\log(\mu_{kj}^{+}) + \log(v_{kj}^{-})\Big), \\
\upsilon_{i,k,j}^{5} &= \log(\mu_{ik}^{+}) - \beta_{ik}\big(\log(\mu_{ik}^{+}) + \log(v_{ik}^{-})\big) + \log(\mu_{kj}^{+}) - \beta_{kj}\Big(\log(\mu_{kj}^{+}) + \log(v_{kj}^{-})\Big), \\
\upsilon_{i,k,j}^{6} &= \log(\mu_{ik}^{+}) - \beta_{ik}\big(\log(\mu_{ik}^{+}) + \log(v_{ik}^{-})\big) + \beta_{jk}\Big(\log(\mu_{jk}^{+}) + \log(v_{jk}^{-})\Big) - \log(\mu_{jk}^{+}).
\end{aligned}$$

Based on formula (24), the following optimization model is established:

$$g^* = \min \sum_{i=+1}^{n-1} \sum_{j=i+1}^{n} \left(\pi_{ij}^+ + \pi_{ij}^- + \iota_{ij}^+ + \iota_{ij}^- + \tau_{ij}^+ + \tau_{ij}^- + \sigma_{ij}^+ + \sigma_{ij}^-\right)$$

$$s.t.\begin{cases} \sum_{k=1}^{i-1} \varsigma_{i,k,j}^1 + \sum_{k=i+1}^{j} \varsigma_{i,k,j}^2 + \sum_{k=j+1}^{n} \varsigma_{i,k,j}^3 = (n-2)\Big(\alpha_{ij}\left(\log(\mu_{ij}^-) + \log(\upsilon_{ij}^+)\right) - \log(\upsilon_{ij}^+)\Big) \\ -\pi_{ij}^+ + \pi_{ij}^-, i,j = 1,2,\ldots,n, i<j \\ \sum_{k=1}^{i-1} \varsigma_{i,k,j}^4 + \sum_{k=i+1}^{j} \varsigma_{i,k,j}^5 + \sum_{k=j+1}^{n} \varsigma_{i,k,j}^6 = (n-2)\Big(\log(\mu_{ij}^-) - \alpha_{ij}\left(\log(\mu_{ij}^-) + \log(\upsilon_{ij}^+)\right)\Big) \\ -\iota_{ij}^+ + \iota_{ij}^-, i,j = 1,2,\ldots,n, i<j \\ \sum_{k=1}^{i-1} \upsilon_{i,k,j}^1 + \sum_{k=i+1}^{j} \upsilon_{i,k,j}^2 + \sum_{k=j+1}^{n} \upsilon_{i,k,j}^3 = (n-2)\Big(\beta_{ij}\left(\log(\mu_{ij}^+) + \log(\upsilon_{ij}^-)\right) - \log(\upsilon_{ij}^-)\Big) \\ -\tau_{ij}^+ + \tau_{ij}^-, i,j = 1,2,\ldots,n, i<j \\ \sum_{k=1}^{i-1} \upsilon_{i,k,j}^4 + \sum_{k=i+1}^{j} \upsilon_{i,k,j}^5 + \sum_{k=j+1}^{n} \upsilon_{i,k,j}^6 = (n-2)\Big(\log(\mu_{ij}^+) - \beta_{ij}\left(\log(\mu_{ij}^+) + \log(\upsilon_{ij}^-)\right)\Big) \\ -\sigma_{ij}^+ + \sigma_{ij}^-, i,j = 1,2,\ldots,n, i<j \\ 1/t \le \mu_{ij}^- \le \mu_{ij}^+ \wedge \mu_{ij}^-\upsilon_{ij}^+ \le 1, \mu_{ij}^- \in U^-, \mu_{ij}^+ \notin U^+, \upsilon_{ij}^+ \notin V^+ \\ \mu_{ij}^- \le \mu_{ij}^+ \le t \wedge \upsilon_{ij}^-\mu_{ij}^+ \le 1, \mu_{ij}^- \notin U^-, \mu_{ij}^+ \in U^+, \upsilon_{ij}^- \notin V^- \\ 1/t \le \upsilon_{ij}^- \le \upsilon_{ij}^+ \wedge \upsilon_{ij}^-\mu_{ij}^+ \le 1, \upsilon_{ij}^- \in V^-, \upsilon_{ij}^+ \notin V^+, \mu_{ij}^+ \notin U^+ \\ \upsilon_{ij}^- \le \upsilon_{ij}^+ \le t \wedge \mu_{ij}^-\upsilon_{ij}^+ \le 1, \upsilon_{ij}^- \notin V^-, \upsilon_{ij}^+ \in V^+, \mu_{ij}^- \notin U^- \\ \begin{cases} 1/t \le \mu_{ij}^- \le \mu_{ij}^+ \le t \wedge \mu_{ij}^-\upsilon_{ij}^+ \le 1 \wedge \mu_{ij}^+\upsilon_{ij}^- \le 1, \\ \mu_{ij}^- \in U^-, \mu_{ij}^+ \in U^+, \upsilon_{ij}^- \notin V^-, \upsilon_{ij}^+ \notin V^+ \end{cases} \\ \begin{cases} 1/t \le \upsilon_{ij}^- \le \upsilon_{ij}^+ \le t \wedge \mu_{ij}^-\upsilon_{ij}^+ \le 1 \wedge \mu_{ij}^+\upsilon_{ij}^- \le 1, \\ \mu_{ij}^- \notin U^-, \mu_{ij}^+ \notin U^+, \upsilon_{ij}^- \in V^-, \upsilon_{ij}^+ \in V^+ \end{cases} \\ \begin{cases} 1/t \le \mu_{ij}^- \le \mu_{ij}^+ \le t \wedge 1/t \le \upsilon_{ij}^- \le \upsilon_{ij}^+ \wedge \mu_{ij}^-\upsilon_{ij}^+ \le 1 \\ \wedge \mu_{ij}^+\upsilon_{ij}^- \le 1, \mu_{ij}^- \in U^-, \mu_{ij}^+ \in U^+, \upsilon_{ij}^- \in V^-, \upsilon_{ij}^+ \notin V^+ \end{cases} \\ \begin{cases} 1/t \le \mu_{ij}^- \le \mu_{ij}^+ \le t \wedge \upsilon_{ij}^- \le \upsilon_{ij}^+ \le t \wedge \mu_{ij}^-\upsilon_{ij}^+ \le 1 \wedge \\ \mu_{ij}^+\upsilon_{ij}^- \le 1, \mu_{ij}^- \in U^-, \mu_{ij}^+ \in U^+, \upsilon_{ij}^- \notin V^-, \upsilon_{ij}^+ \in V^+ \end{cases} \\ \begin{cases} 1/t \le \mu_{ij}^- \le \mu_{ij}^+ \wedge 1/t \le \upsilon_{ij}^- \le \upsilon_{ij}^+ \le t \wedge \mu_{ij}^-\upsilon_{ij}^+ \le 1 \wedge \\ \mu_{ij}^+\upsilon_{ij}^- \le 1, \mu_{ij}^- \in U^-, \mu_{ij}^+ \notin U^+, \upsilon_{ij}^- \in V^-, \upsilon_{ij}^+ \in V^+ \end{cases} \\ \begin{cases} \mu_{ij}^- \le \mu_{ij}^+ \le t \wedge 1/t \le \upsilon_{ij}^- \le \upsilon_{ij}^+ \le t \wedge \mu_{ij}^-\upsilon_{ij}^+ \le 1 \wedge \\ \mu_{ij}^+\upsilon_{ij}^- \le 1, \mu_{ij}^- \notin U^-, \mu_{ij}^+ \in U^+, \upsilon_{ij}^- \in V^-, \upsilon_{ij}^+ \in V^+ \end{cases} \\ \begin{cases} 1/t \le \mu_{ij}^- \le \mu_{ij}^+ \le t \wedge 1/t \le \upsilon_{ij}^- \le \upsilon_{ij}^+ \le t \wedge \mu_{ij}^-\upsilon_{ij}^+ \le \\ 1 \wedge \mu_{ij}^+\upsilon_{ij}^- \le 1, \mu_{ij}^- \in U^-, \mu_{ij}^+ \in U^+, \upsilon_{ij}^- \in V^-, \upsilon_{ij}^+ \in V^+ \end{cases} \\ \alpha_{ij} + \alpha_{ji} = 1, \beta_{ij} + \beta_{ji} = 1, \alpha_{ij} = 0 \vee 1, \\ \beta_{ij} = 0 \vee 1, i,j = 1,2,\ldots,n, i<j \\ \pi_{ij}^+, \pi_{ij}^-, \iota_{ij}^+, \iota_{ij}^-, \tau_{ij}^+, \tau_{ij}^-, \sigma_{ij}^+, \sigma_{ij}^- \ge 0, i,j = 1,2,\ldots,n, i<j \end{cases} \tag{M-2}$$

where

$$U^- = \left\{\mu_{ij}^- | s_{\mu_{ij}^-} \text{is missing}, \ i, \ j = 1, 2, \ldots, n, \ i < j\right\},$$
$$U^+ = \left\{\mu_{ij}^+ | s_{\mu_{ij}^+} \text{is missing}, \ i, \ j = 1, 2, \ldots, n, \ i < j\right\},$$
$$V^- = \left\{\upsilon_{ij}^- | s_{\upsilon_{ij}^-} \text{is missing}, \ i, \ j = 1, 2, \ldots, n, \ i < j\right\},$$
$$V^+ = \left\{\upsilon_{ij}^+ | s_{\upsilon_{ij}^+} \text{is missing}, \ i, \ j = 1, 2, \ldots, n, \ i < j\right\},$$

and t is the biggest index of linguistic variables in the CALTS S'.

Example 1 Let $X = \{x_1, x_2, x_3, x_4\}$ be the set of objects, and let $\tilde{R} = (\tilde{r}_{ij})_{4\times 4}$ be an incomplete IVIMLPR on X for the CALTS $S' = \{s_b | 1/4 \le b \le 4\}$, where

$$\tilde{R} = \begin{pmatrix} ([s_1, s_1], [s_1, s_1]) & \left([s_{1/3}, s_{1/2}], [s_{v_{12}^-}, s_{v_{12}^+}]\right) & ([s_1, s_3], [s_{1/4}, s_{1/2}]) & ([s_1, s_2], [s_{1/3}, s_{1/2}]) \\ \left([s_{\mu_{21}^-}, s_{\mu_{21}^+}], [s_{1/3}, s_{1/2}]\right) & ([s_1, s_1], [s_1, s_1]) & \left([s_{\mu_{23}^-}, s_{\mu_{23}^+}], [s_{v_{23}^-}, s_{v_{23}^+}]\right) & \left([s_{\mu_{24}^-}, s_{\mu_{24}^+}], [s_{1/3}, s_1]\right) \\ ([s_{1/4}, s_{1/2}], [s_1, s_3]) & \left([s_{\mu_{32}^-}, s_{\mu_{32}^+}], [s_{v_{32}^-}, s_{v_{32}^+}]\right) & ([s_1, s_1], [s_1, s_1]) & ([s_{1/2}, s_1], [s_{1/2}, s_2]) \\ ([s_{1/3}, s_{1/2}], [s_1, s_2]) & \left([s_{1/3}, s_1], [s_{v_{24}^-}, s_{v_{24}^+}]\right) & ([s_{1/2}, s_2], [s_{1/2}, s_1]) & ([s_1, s_1], [s_1, s_1]) \end{pmatrix}$$

Using model (M-2), unknown linguistic variables are determined as follows:

$$[s_{v_{12}^-}, s_{v_{12}^+}] = [s_{0.63}, s_{0.63}],\ [s_{\mu_{24}^-}, s_{\mu_{24}^+}] = [s_1, s_{2.62}]\left([s_{\mu_{23}^-}, s_{\mu_{23}^+}], [s_{v_{23}^-}, s_{v_{23}^+}]\right) = ([s_{1.59}, s_{1.82}], [s_{0.25}, s_{0.41}]).$$

5.2 Inconsistent IVIMLPRs

In general, preference relations provided by DMs are inconsistent. It is indispensable to adjust the consistency of preference relations for ranking objects reasonably. Therefore, this subsection studies inconsistent IVIMLPRs. Based on consistency analysis, several optimization models are established for adjusting inconsistent IVIMLPRs and deriving completely consistent IVIMLPRs.

Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an any given IVIMLPR. If $f^* \neq 0$ following model (M-1), then $\tilde{R}$ is inconsistent. In this case, we build the following optimization model for determining QMILPRs with the highest consistency level:

$$h^* = \max \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\alpha_{ij} + \beta_{ij}\right)$$

$$s.t.\begin{cases} \left(\alpha_{ik}\left(\log(\mu_{ik}^-) + \log(v_{ik}^+)\right) - \log(v_{ik}^+)\right) + \left(\alpha_{kj}\left(\log(\mu_{kj}^-) + \log(v_{kj}^+)\right)\right. \\ \left. - \log(v_{kj}^+)\right) = \alpha_{ij}\left(\log(\mu_{ij}^-) + \log(v_{ij}^+)\right) - \log(v_{ij}^+) - \varepsilon_{k,ij}^+ + \varepsilon_{k,ij}^- \\ \left(\log(\mu_{ik}^-) - \alpha_{ik}\left(\log(v_{ik}^+) + \log(\mu_{ik}^-)\right)\right) + \left(\log(\mu_{kj}^-) - \alpha_{kj}\left(\log(v_{kj}^+) + \right.\right. \\ \left.\left. + \log(\mu_{kj}^-)\right)\right) = \log(\mu_{ij}^-) - \alpha_{ij}\left(\log(v_{ij}^+) + \log(\mu_{ij}^-)\right) - \delta_{k,ij}^+ + \delta_{k,ij}^- \\ \left(\beta_{ik}\left(\log(\mu_{ik}^+) + \log(v_{ik}^-)\right) - \log(v_{ik}^-)\right) + \left(\beta_{kj}\left(\log(\mu_{kj}^+) + \log(v_{kj}^-)\right)\right. \\ \left. - \log(v_{kj}^-)\right) = \beta_{ij}\left(\log(\mu_{ij}^+) + \log(v_{ij}^-)\right) - \log(v_{ij}^-) - \theta_{k,ij}^+ + \theta_{k,ij}^- \\ \left(\log(\mu_{ik}^+) - \beta_{ik}\left(\log(v_{ik}^-) + \log(\mu_{ik}^+)\right)\right) + \left(\log(\mu_{kj}^+) - \beta_{kj}\left(\log(v_{kj}^-) + \right.\right. \\ \left.\left. + \log(\mu_{kj}^+)\right)\right) = \log(\mu_{ij}^+) - \left(\beta_{ij}\log(v_{ij}^-) + \log(\mu_{ij}^+)\right) - \vartheta_{k,ij}^+ + \vartheta_{k,ij}^- \\ \sum_{i=1}^{n-2} \sum_{k=i+1}^{n-1} \sum_{j=k+1}^{n} \begin{pmatrix} \varepsilon_{k,ij}^+ + \varepsilon_{k,ij}^- + \delta_{k,ij}^+ + \delta_{k,ij}^- \\ +\theta_{k,ij}^+ + \theta_{k,ij}^- + \vartheta_{k,ij}^+ + \vartheta_{k,ij}^- \end{pmatrix} = f^* \\ \alpha_{ij} = 0 \vee 1, \beta_{ij} = 0 \vee 1, i, j = 1, 2, \ldots, n, i < j \\ \varepsilon_{k,ij}^+, \varepsilon_{k,ij}^-, \delta_{k,ij}^+, \delta_{k,ij}^-, \theta_{k,ij}^+, \theta_{k,ij}^-, \vartheta_{k,ij}^+, \vartheta_{k,ij}^- \ge 0, \\ i, k, j = 1, 2, \ldots, n, i < k < j \end{cases} \tag{M-3}$$

By solving model (M-3), the optimal 0–1 indicator variables α_{ij}^* and β_{ij}^* can be obtained, where $i, j = 1, 2,\ldots, n$ such that $i < j$. Following the derived 0–1 indicator variables, the QMILPRs $\bar{P}_1 = (\bar{p}_{1,ij})_{n\times n}$ and $\bar{P}_2 = (\bar{p}_{2,ij})_{n\times n}$ are derived having the highest consistency level.

Let $\bar{P}_1 = (\bar{p}_{1,ij})_{n\times n} = \left([p_{1,ij}^l, p_{1,ij}^u]\right)_{n\times n}$ and $\bar{P}_2 = (\bar{p}_{2,ij})_{n\times n}$ $=([p_{2,ij}^l, p_{2,ij}^u])_{n\times n}$. Next, we adjust the consistency of $\bar{P}_1$ and $\bar{P}_2$ for deriving the consistent IVIMLPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$. Let

$$I(\bar{P}_1) = (I(\bar{p}_{1,ij}))_{n\times n} = \left([I(p_{1,ij}^l), I(p_{1,ij}^u)]\right)_{n\times n} I(\bar{P}_2)$$
$$= (I(\bar{p}_{2,ij}))_{n\times n} = \left([I(p_{2,ij}^l), I(p_{2,ij}^u)]\right)_{n\times n}$$

For per (i, j) such that $i < j$, let $s_{1,ij}^l$ and $s_{1,ij}^u$ be the respective left and right adjustments of $\bar{p}_{1,ij}$'s endpoints, and let $s_{2,ij}^l$ and $s_{2,ij}^u$ be the respective left and right adjustments of $\bar{p}_{2,ij}$'s endpoints, where $1/t \le s_{1,ij}^l, s_{1,ij}^u, s_{2,ij}^l, s_{2,ij}^u \le t$. Then,

$$\begin{cases} [p_{1,ij}^l \otimes s_{1,ij}^l, p_{1,ij}^u \otimes s_{1,ij}^u] = [p_{1,ik}^l \otimes s_{1,ik}^l, p_{1,ik}^u \otimes s_{1,ik}^u] \otimes [p_{1,kj}^l \otimes s_{1,kj}^l, p_{1,kj}^u \otimes s_{1,kj}^u] \\ [p_{2,ij}^l \otimes s_{2,ij}^l, p_{2,ij}^u \otimes s_{2,ij}^u] = [p_{2,ik}^l \otimes s_{2,ik}^l, p_{2,ik}^u \otimes s_{2,ik}^u] \otimes [p_{2,kj}^l \otimes s_{2,kj}^l, p_{2,kj}^u \otimes s_{2,kj}^u] \end{cases} \tag{25}$$

for each triple of (i, k, j) such that $i < k < j$.

Using (25), we obtain

$$\begin{cases} I(p_{1,ij}^l) \times I(s_{1,ij}^l) = I(p_{1,ik}^l) \times I(s_{1,ik}^l) \times I(p_{1,kj}^l) \times I(s_{1,kj}^l) \\ I(p_{1,ij}^u) \times I(s_{1,ij}^u) = I(p_{1,ik}^u) \times I(s_{1,ik}^u) \times I(p_{1,kj}^u) \times I(s_{1,kj}^u) \\ I(p_{2,ij}^l) \times I(s_{2,ij}^l) = I(p_{2,ik}^l) \times I(s_{2,ik}^l) \times I(p_{2,kj}^l) \times I(s_{2,kj}^l) \\ I(p_{2,ij}^u) \times I(s_{2,ij}^u) = I(p_{2,ik}^u) \times I(s_{2,ik}^u) \times I(p_{2,kj}^u) \times I(s_{2,kj}^u) \end{cases} \tag{26}$$

Taking the logarithm for formula (26), we have

$$\begin{cases} \log\left(I(p_{1,ij}^l)\right) + \log\left(I(s_{1,ij}^l)\right) = \log\left(I(p_{1,ik}^l)\right) + \log\left(I(s_{1,ik}^l)\right) + \log\left(I(p_{1,kj}^l)\right) + \log\left(I(s_{1,kj}^l)\right) \\ \log\left(I(p_{1,ij}^u)\right) + \log\left(I(s_{1,ij}^u)\right) = \log(I(p_{1,ik}^u)) + \log(I(s_{1,ik}^u)) + \log\left(I(p_{1,kj}^u)\right) + \log\left(I(s_{1,kj}^u)\right) \\ \log\left(I(p_{2,ij}^l)\right) + \log\left(I(s_{2,ij}^l)\right) = \log\left(I(p_{2,ik}^l)\right) + \log\left(I(s_{2,ik}^l)\right) + \log\left(I(p_{2,kj}^l)\right) + \log\left(I(s_{2,kj}^l)\right) \\ \log\left(I(p_{2,ij}^u)\right) + \log\left(I(s_{2,ij}^u)\right) = \log(I(p_{2,ik}^u)) + \log(I(s_{2,ik}^u)) + \log\left(I(p_{2,kj}^u)\right) + \log\left(I(s_{2,kj}^u)\right) \end{cases} \tag{27}$$

However, formula (27) cannot guarantee the adjusted linguistic variables fall into $S' = \{s_b | b \in [1/t,\, t]\}$, namely, we have

$$\begin{cases} -\log(t) \le \log\Big(I(p^l_{1,ij})\Big) + \log\Big(I(s^l_{1,ij})\Big) \le \log(t) \\ -\log(t) \le \log\Big(I(p^u_{1,ij})\Big) + \log\Big(I(s^u_{1,ij})\Big) \le \log(t) \\ -\log(t) \le \log\Big(I(p^l_{2,ij})\Big) + \log\Big(I(s^l_{2,ij})\Big) \le \log(t) \\ -\log(t) \le \log\Big(I(p^u_{2,ij})\Big) + \log\Big(I(s^u_{2,ij})\Big) \le \log(t) \end{cases} \tag{28}$$

for per (i, j) such that $i < j$.

In addition, to retain information offered by the DMs, the adjustment should be as small as possible. Thus, we construct the following optimization model to adjust the QMILPRs $\bar{P}_1$ and $\bar{P}_2$:

$$\varphi^* = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Big(\log\Big(I(s^l_{1,ij})\Big) + \log\Big(I(s^u_{1,ij})\Big) + \log\Big(I(s^l_{2,ij})\Big) + \log\Big(I(s^u_{2,ij})\Big)\Big)$$

$$s.t. \begin{cases} \log\Big(I(p^l_{1,ij})\Big) + \log\Big(I(s^l_{1,ij})\Big) = \log\Big(I(p^l_{1,ik})\Big) + \log\Big(I(s^l_{1,ik})\Big) + \log\Big(I(p^l_{1,kj})\Big) \\ \qquad + \log\Big(I(s^l_{1,kj})\Big) \\ \log\Big(I(p^u_{1,ij})\Big) + \log\Big(I(s^u_{1,ij})\Big) = \log\big(I(p^u_{1,ik})\big) + \log\big(I(s^u_{1,ik})\big) + \log\Big(I(p^u_{1,kj})\Big) \\ \qquad + \log\Big(I(s^u_{1,kj})\Big) \\ \log\Big(I(p^l_{2,ij})\Big) + \log\Big(I(s^l_{2,ij})\Big) = \log\Big(I(p^l_{2,ik})\Big) + \log\Big(I(s^l_{2,ik})\Big) + \log\Big(I(p^l_{2,kj})\Big) \\ \qquad + \log\Big(I(s^l_{2,kj})\Big) \\ \log\Big(I(p^u_{2,ij})\Big) + \log\Big(I(s^u_{2,ij})\Big) = \log\big(I(p^u_{2,ik})\big) + \log\big(I(s^u_{2,ik})\big) + \log\Big(I(p^u_{2,kj})\Big) \\ \qquad + \log\Big(I(s^u_{2,kj})\Big) \\ i, k, j = 1, 2, \ldots, n, i < k < j \\ -\log(t) \le \log\Big(I(p^l_{1,ij})\Big) + \log\Big(I(s^l_{1,ij})\Big) \le \log(t) \\ -\log(t) \le \log\Big(I(p^u_{1,ij})\Big) + \log\Big(I(s^u_{1,ij})\Big) \le \log(t) \\ -\log(t) \le \log\Big(I(p^l_{2,ij})\Big) + \log\Big(I(s^l_{2,ij})\Big) \le \log(t) \\ -\log(t) \le \log\Big(I(p^u_{2,ij})\Big) + \log\Big(I(s^u_{2,ij})\Big) \le \log(t), \\ i, j = 1, 2, \ldots, n, i < j \end{cases} \tag{M-4}$$

By solving model (M-4), we get completely consistent QMILPRs, and then the associated completely consistent IVIMLPRs can be derived. However, the completely consistent QMILPRs obtained from model (M-4) cannot guarantee the conditions of elements in IVIMLPRs, namely, one of the four following cases is true

$$
\begin{aligned}
&(1)\ \begin{cases} p^l_{1,ij} \otimes s^l_{1,ij} \le p^u_{1,ij} \otimes s^u_{1,ij} \\ p^l_{2,ij} \otimes s^l_{2,ij} \le p^u_{2,ij} \otimes s^u_{2,ij} \end{cases} \Rightarrow \begin{cases} p^l_{1,ij} \otimes s^l_{1,ij} \le p^l_{2,ij} \otimes s^l_{2,ij} \\ \left(p^u_{2,ij} \otimes s^u_{2,ij}\right)^{-1} \le \left(p^u_{1,ij} \otimes s^u_{1,ij}\right)^{-1} \end{cases} \\
&(2)\ \begin{cases} p^l_{1,ij} \otimes s^l_{1,ij} > p^u_{1,ij} \otimes s^u_{1,ij} \\ p^l_{2,ij} \otimes s^l_{2,ij} \le p^u_{2,ij} \otimes s^u_{2,ij} \end{cases} \Rightarrow \begin{cases} p^u_{1,ij} \otimes s^u_{1,ij} \le p^l_{2,ij} \otimes s^l_{2,ij} \\ \left(p^u_{2,ij} \otimes s^u_{2,ij}\right)^{-1} \le \left(p^l_{1,ij} \otimes s^l_{1,ij}\right)^{-1} \end{cases} \\
&(3)\ \begin{cases} p^l_{1,ij} \otimes s^l_{1,ij} \le p^u_{1,ij} \otimes s^u_{1,ij} \\ p^l_{2,ij} \otimes s^l_{2,ij} > p^u_{2,ij} \otimes s^u_{2,ij} \end{cases} \Rightarrow \begin{cases} p^l_{1,ij} \otimes s^l_{1,ij} \le p^u_{2,ij} \otimes s^u_{2,ij} \\ \left(p^l_{2,ij} \otimes s^l_{2,ij}\right)^{-1} \le \left(p^u_{1,ij} \otimes s^u_{1,ij}\right)^{-1} \end{cases} \\
&(4)\ \begin{cases} p^l_{1,ij} \otimes s^l_{1,ij} > p^u_{1,ij} \otimes s^u_{1,ij} \\ p^l_{2,ij} \otimes s^l_{2,ij} > p^u_{2,ij} \otimes s^u_{2,ij} \end{cases} \Rightarrow \begin{cases} p^u_{1,ij} \otimes s^u_{1,ij} \le p^u_{2,ij} \otimes s^u_{2,ij} \\ \left(p^l_{2,ij} \otimes s^l_{2,ij}\right)^{-1} \le \left(p^l_{1,ij} \otimes s^l_{1,ij}\right)^{-1} \end{cases}
\end{aligned}
$$

for per (i, j) such that $i < j$.

To solve this issue, we further build the following optimization model based on model (M-4):

$$
\psi^* = \min \sum\nolimits_{i=1}^{n-1} \sum\nolimits_{j=i+1}^{n} \left(\log\left(I(s^l_{1,ij})\right) + \log\left(I(s^u_{1,ij})\right) + \log\left(I(s^l_{2,ij})\right) + \log\left(I(s^u_{2,ij})\right)\right)
$$

$$
s.t. \begin{cases}
\log\left(I(p^l_{1,ij})\right) + \log\left(I(s^l_{1,ij})\right) = \log\left(I(p^l_{1,ik})\right) + \log\left(I(s^l_{1,ik})\right) + \log\left(I(p^l_{1,kj})\right) + \log\left(I(s^l_{1,kj})\right) \\
\log\left(I(p^u_{1,ij})\right) + \log\left(I(s^u_{1,ij})\right) = \log\left(I(p^u_{1,ik})\right) + \log\left(I(s^u_{1,ik})\right) + \log\left(I(p^u_{1,kj})\right) + \log\left(I(s^u_{1,kj})\right) \\
\log\left(I(p^l_{2,ij})\right) + \log\left(I(s^l_{2,ij})\right) = \log\left(I(p^l_{2,ik})\right) + \log\left(I(s^l_{2,ik})\right) + \log\left(I(p^l_{2,kj})\right) + \log\left(I(s^l_{2,kj})\right) \\
\log\left(I(p^u_{2,ij})\right) + \log\left(I(s^u_{2,ij})\right) = \log\left(I(p^u_{2,ik})\right) + \log\left(I(s^u_{2,ik})\right) + \log\left(I(p^u_{2,kj})\right) + \log\left(I(s^u_{2,kj})\right) \\
i, k, j = 1, 2, \ldots, n, i < k < j \\
-\log(t) \le \log\left(I(p^l_{1,ij})\right) + \log\left(I(s^l_{1,ij})\right) \le \log(t) \\
-\log(t) \le \log\left(I(p^u_{1,ij})\right) + \log\left(I(s^u_{1,ij})\right) \le \log(t) \\
-\log(t) \le \log\left(I(p^l_{2,ij})\right) + \log\left(I(s^l_{2,ij})\right) \le \log(t) \\
-\log(t) \le \log\left(I(p^u_{2,ij})\right) + \log\left(I(s^u_{2,ij})\right) \le \log(t) \\
\chi^1_{ij} I^1_{ij} \wedge \chi^1_{ij} I^2_{ij} \wedge \chi^1_{ij} I^3_{ij} \wedge \chi^1_{ij} I^4_{ij} \le 0 \\
\chi^2_{ij} I^1_{ij} > 0, \chi^2_{ij} I^2_{ij} \wedge \chi^2_{ij} I^5_{ij} \wedge \chi^2_{ij} I^6_{ij} \le 0 \\
\chi^3_{ij} I^2_{ij} > 0, \chi^3_{ij} I^1_{ij} \wedge \chi^3_{ij} I^5_{ij} \wedge \chi^3_{ij} I^6_{ij} \le 0 \\
\chi^4_{ij} I^1_{ij} \wedge \chi^4_{ij} I^2_{ij} > 0, \chi^4_{ij} I^5_{ij} \wedge \chi^4_{ij} I^6_{ij} \le 0 \\
i, j = 1, 2, \ldots, n, i < j \\
\chi^1_{ij}, \chi^2_{ij}, \chi^3_{ij}, \chi^4_{ij} = 0 \vee 1, i, j = 1, 2, \ldots, n, i < j \\
\chi^1_{ij} + \chi^2_{ij} + \chi^3_{ij} + \chi^4_{ij} = 1, i, j = 1, 2, \ldots, n, i < j
\end{cases}
\tag{M-5}
$$

where

$$\begin{cases} I_{ij}^{1} = \log\left(I(p_{1,ij}^{l})\right) + \log\left(I(s_{1,ij}^{l})\right) - \log\left(I(p_{1,ij}^{u})\right) - \log\left(I(s_{1,ij}^{u})\right) \\ I_{ij}^{2} = \log\left(I(p_{2,ij}^{l})\right) + \log\left(I(s_{2,ij}^{l})\right) - \log\left(I(p_{2,ij}^{u})\right) - \log\left(I(s_{2,ij}^{u})\right) \\ I_{ij}^{3} = \log\left(I(p_{1,ij}^{l})\right) + \log\left(I(s_{1,ij}^{l})\right) - \log\left(I(p_{2,ij}^{l})\right) - \log\left(I(s_{2,ij}^{l})\right) \\ I_{ij}^{4} = \log\left(I(p_{1,ij}^{u})\right) + \log\left(I(s_{1,ij}^{u})\right) - \log\left(I(p_{2,ij}^{u})\right) - \log\left(I(s_{2,ij}^{u})\right) \\ I_{ij}^{5} = \log\left(I(p_{1,ij}^{u})\right) + \log\left(I(s_{1,ij}^{u})\right) - \log\left(I(p_{2,ij}^{l})\right) - \log\left(I(s_{2,ij}^{l})\right) \\ I_{ij}^{6} = \log\left(I(p_{1,ij}^{l})\right) + \log\left(I(s_{1,ij}^{l})\right) - \log\left(I(p_{2,ij}^{u})\right) - \log\left(I(s_{2,ij}^{u})\right) \end{cases}.$$

By solving model (M-5), we can derive the completely consistent QMILPRs $\bar{Q}_1^* = (\bar{q}_{1,ij}^*)_{n\times n}$ and $\bar{Q}_2^*= (\bar{q}_{2,ij}^*)_{n\times n}$, where $\bar{q}_{1,ij}^* = \left[p_{1,ij}^l \otimes s_{1,ij}^{*l},\, p_{1,ij}^u \otimes s_{1,ij}^{*u}\right]$, and $\bar{q}_{1,ij}^* = \left[p_{1,ij}^l \otimes s_{1,ij}^{*l},\, p_{1,ij}^u \otimes s_{1,ij}^{*u}\right]$ for all $i, j = 1, 2, \ldots, n$. Based on the relationships between QMILPRs and IVIMLPRs, we can derive the completely consistent IVIMLPR $\tilde{R}^* = (\tilde{r}_{ij}^*)_{n\times n} = \left([s_{\mu_{ij}^-}^*, s_{\mu_{ij}^+}^*], [s_{v_{ij}^-}^*, s_{v_{ij}^+}^*]\right)_{n\times n}$, where

$$\begin{cases} p_{1,ij}^l \otimes s_{1,ij}^{*l} \le p_{1,ij}^u \otimes s_{1,ij}^{*u} \\ p_{2,ij}^l \otimes s_{2,ij}^{*l} \le p_{2,ij}^u \otimes s_{2,ij}^{*u} \end{cases} \Rightarrow \begin{cases} s_{\mu_{ij}^-}^* = p_{1,ij}^l \otimes s_{1,ij}^{*l},\ s_{v_{ij}^+}^* = \left(p_{1,ij}^u \otimes s_{1,ij}^{*u}\right)^{-1} \\ s_{\mu_{ij}^+}^* = p_{2,ij}^l \otimes s_{2,ij}^{*l},\ s_{v_{ij}^-}^* = \left(p_{2,ij}^u \otimes s_{2,ij}^{u}\right)^{-1} \end{cases}$$

$$\begin{cases} p_{1,ij}^l \otimes s_{1,ij}^{*l} > p_{1,ij}^u \otimes s_{1,ij}^{*u} \\ p_{2,ij}^l \otimes s_{2,ij}^{*l} \le p_{2,ij}^u \otimes s_{2,ij}^{*u} \end{cases} \Rightarrow \begin{cases} s_{\mu_{ij}^-}^* = p_{1,ij}^u \otimes s_{1,ij}^{*u},\ s_{v_{ij}^+}^* = \left(p_{1,ij}^l \otimes s_{1,ij}^{*l}\right)^{-1} \\ s_{\mu_{ij}^+}^* = p_{2,ij}^l \otimes s_{2,ij}^{*l},\ s_{v_{ij}^-}^* = \left(p_{2,ij}^u \otimes s_{2,ij}^{u}\right)^{-1} \end{cases}$$

$$\begin{cases} p_{1,ij}^l \otimes s_{1,ij}^{*l} \le p_{1,ij}^u \otimes s_{1,ij}^{*u} \\ p_{2,ij}^l \otimes s_{2,ij}^{*l} > p_{2,ij}^u \otimes s_{2,ij}^{*u} \end{cases} \Rightarrow \begin{cases} s_{\mu_{ij}^-}^* = p_{1,ij}^l \otimes s_{1,ij}^{*l},\ s_{v_{ij}^+}^* = \left(p_{1,ij}^u \otimes s_{1,ij}^{*u}\right)^{-1} \\ s_{\mu_{ij}^+}^* = p_{2,ij}^u \otimes s_{2,ij}^{u},\ s_{v_{ij}^-}^* = \left(p_{2,ij}^l \otimes s_{2,ij}^{*l}\right)^{-1} \end{cases}$$

$$\begin{cases} p_{1,ij}^l \otimes s_{1,ij}^{*l} > p_{1,ij}^u \otimes s_{1,ij}^{*u} \\ p_{2,ij}^l \otimes s_{2,ij}^{*l} > p_{2,ij}^u \otimes s_{2,ij}^{*u} \end{cases} \Rightarrow \begin{cases} s_{\mu_{ij}^-}^* = p_{1,ij}^u \otimes s_{1,ij}^{*u},\ s_{v_{ij}^+}^* = \left(p_{1,ij}^l \otimes s_{1,ij}^{*l}\right)^{-1} \\ s_{\mu_{ij}^+}^* = p_{2,ij}^u \otimes s_{2,ij}^{u},\ s_{v_{ij}^-}^* = \left(p_{2,ij}^l \otimes s_{2,ij}^{*l}\right)^{-1} \end{cases}$$

for all $i, j = 1, 2, \ldots, n$ such that $i < j$.

For the derived complete IVIMLPR $\tilde{R} = (\tilde{r}_{ij})_{4\times 4}$ shown in Example 1, we get $f^* = 6.9315$ following model (M-1). Thus, this complete IVIMLPR $\tilde{R}$ is inconsistent. Based on model (M-3), the optimal 0–1 indicator variables are derived as follows:

$$\begin{cases} \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{24} = \alpha_{34} = 1,\ \alpha_{23} = 0 \\ \beta_{12} = \beta_{13} = \beta_{14} = \beta_{34} = 1,\ \beta_{23} = \beta_{24} = 0 \end{cases}$$

by which the following QMILPRs are derived:

$$\bar{Q}_1 = \begin{pmatrix} [s_1, s_1] & [s_{0.33}, s_{0.5}] & [s_1, s_3] & [s_1, s_2] \\ [s_3, s_2] & [s_1, s_1] & [s_{1.82}, s_{1.59}] & [s_1, s_{2.62}] \\ [s_1, s_{0.33}] & [s_{0.55}, s_{0.63}] & [s_1, s_1] & [s_{0.5}, s_1] \\ [s_1, s_{0.5}] & [s_1, s_{0.38}] & [s_2, s_1] & [s_1, s_1] \end{pmatrix}$$

$$\bar{Q}_2 = \begin{pmatrix} [s_1, s_1] & [s_{0.63}, s_{0.63}] & [s_{0.25}, s_{0.5}] & [s_{0.33}, s_{0.5}] \\ [s_{1.59}, s_{1.59}] & [s_1, s_1] & [s_{0.25}, s_{0.41}] & [s_{0.33}, s_1] \\ [s_4, s_2] & [s_4, s_{2.44}] & [s_1, s_1] & [s_{0.5}, s_2] \\ [s_3, s_2] & [s_3, s_1] & [s_2, s_{0.5}] & [s_1, s_1] \end{pmatrix}$$

For these two QMILPRs, the following completely consistent QMILPRs are obtained using model (M-5):

$$\bar{Q}_1^* = \begin{pmatrix} [s_1, s_1] & [s_1, s_{1.15}] & [s_{1.82}, s_3] & [s_1, s_3] \\ [s_1, s_{0.87}] & [s_1, s_1] & [s_{1.82}, s_{2.62}] & [s_1, s_{2.62}] \\ [s_{0.55}, s_{0.33}] & [s_{0.55}, s_{0.38}] & [s_1, s_1] & [s_{0.55}, s_1] \\ [s_1, s_{0.33}] & [s_1, s_{0.38}] & [s_{1.82}, s_1] & [s_1, s_1] \end{pmatrix}$$

$$\bar{Q}_2^* = \begin{pmatrix} [s_1, s_1] & [s_{1.15}, s_1] & [s_3, s_{1.82}] & [s_{1.65}, s_{3.64}] \\ [s_{0.87}, s_1] & [s_1, s_1] & [s_{2.62}, s_{1.82}] & [s_{1.44}, s_{3.64}] \\ [s_{0.33}, s_{0.55}] & [s_{0.38}, s_{0.55}] & [s_1, s_1] & [s_{0.55}, s_2] \\ [s_{0.61}, s_{0.27}] & [s_{0.69}, s_{0.27}] & [s_{1.82}, s_{0.5}] & [s_1, s_1] \end{pmatrix}$$

Then, the corresponding consistent IVIMLPR is

$$\tilde{R}^* = \begin{pmatrix} ([s_1, s_1], [s_1, s_1]) & ([s_1, s_1], [s_{0.87}, s_{0.87}]) & ([s_{1.82}, s_{1.82}], [s_{0.33}, s_{0.33}]) & ([s_1, s_{1.65}], [s_{0.27}, s_{0.33}]) \\ ([s_{0.87}, s_{0.87}], [s_1, s_1]) & ([s_1, s_1], [s_1, s_1]) & ([s_{1.82}, s_{1.82}], [s_{0.38}, s_{0.38}]) & ([s_1, s_{1.44}], [s_{0.27}, s_{0.38}]) \\ ([s_{0.33}, s_{0.33}], [s_{1.82}, s_{1.82}]) & ([s_{0.38}, s_{0.38}], [s_{1.82}, s_{1.82}]) & ([s_1, s_1], [s_1, s_1]) & ([s_{0.55}, s_{0.55}], [s_{0.5}, s_1]) \\ ([s_{0.27}, s_{0.33}], [s_1, s_{1.65}]) & ([s_{0.27}, s_{0.38}], [s_1, s_{1.44}]) & ([s_{0.5}, s_1], [s_{0.55}, s_{0.55}]) & ([s_1, s_1], [s_1, s_1]) \end{pmatrix}$$

6 GDM with IVIMLPRs

To derive the objective ranking of objects, more than one DM is usually needed, namely, the so-called GDM. This section studies GDM with IVIMLPRs. To do this, the section contains two parts. The first part studies consensus that is a necessary step for measuring the agreement degree of DMs' preferences for final ranking (Cabrerizo et al. 2015; del Moral et al. 2018; Dong et al. 2019; Herrera-Viedma et al. 2014; Liu et al. 2017; Liu et al. 2019a, b, c, d; Perez et al. 2014; Wu et al. 2019b, c; Zhang et al. 2018a, b), and the second part offers a method for GDM with IVIMLPRs.

6.1 Consensus Analysis

Considering a GDM problem, suppose that there are m DMs, namely, $E = \{e_1, e_2, \ldots, e_m\}$, who are invited to compare objects in $X = \{x_1, x_2, \ldots, x_n\}$ for the CALTS $S' = \{s_b | b \in [1/t, t]\}$. Let $\tilde{R}^k = (\tilde{r}_{ij}^k)_{n\times n}$ be the individual IVIMLPR provided by the

DM e_k, $k = 1, 2, \ldots, m$, where $\tilde{r}_{ij}^k = \left([s^k_{\mu^-_{ij}}, s^k_{\mu^+_{ij}}], [s^k_{v^-_{ij}}, s^k_{v^+_{ij}}]\right)$ is the IVIMLV offered by the DM e_k for the object x_j over x_j, $i, j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$.

Definition 15 Let $\tilde{R}^k = (\tilde{r}^k_{ij})_{n\times n}$, $k = 1, 2, \ldots, m$, be any given m IVIMLPRs, and let $\bar{Q}_1^{*,k} = (\bar{q}^{*,k}_{1,ij})_{n\times n}$ and $\bar{Q}_2^{*,k} = (\bar{q}^{*,k}_{2,ij})_{n\times n}$ be the associated individual QMILPRs. Then, the comprehensive QMILPRs $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ and $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$ are defined as:

$$\begin{cases} \bar{q}_{1,ij} = \otimes_{k=1}^m \left(\bar{q}^{*,k}_{1,ij}\right)^{w_k} \\ \bar{q}_{2,ij} = \otimes_{k=1}^m \left(\bar{q}^{*,k}_{2,ij}\right)^{w_k} \end{cases} \tag{29}$$

where $w = (w_1, w_2, \ldots, w_m)$ is a weight vector such that $\sum_{k=1}^m w_k = 1$ and $w_k \geq 0$ for all $k = 1, 2, \ldots, m$.

Remark 4 Similar to the analysis for model (M-4), the comprehensive QMILPRs derived from formula (29) may not satisfy one of the four cases (1)–(2). Therefore, model (M-5) is adopted to adjust comprehensive QMILPRs $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ and $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$. Note that the adjusted comprehensive QMILPRs are completely consistent.

Property 2 *Let $\tilde{R}^k = (\tilde{r}^k_{ij})_{n\times n}$, $k = 1, 2, \ldots, m$, be any given m IVIMLPRs, and let $\bar{Q}_1^{*,k} = (\bar{q}^{*,k}_{1,ij})_{n\times n}$ and $\bar{Q}_2^{*,k} = (\bar{q}^{*,k}_{2,ij})_{n\times n}$ be the associated individual consistent QMILPRs. Then, the comprehensive QMILPRs $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ and $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$ obtained from formula (29) are completely consistent.*

Proof For per (i, j), following formula (29) we get

$$\begin{aligned} \bar{q}_{1,ij} &= \otimes_{k=1}^m \left(\bar{q}^{*,k}_{1,ij}\right)^{w_k} = \otimes_{l=1}^m \left(\bar{q}^{*,k}_{1,il} \otimes \bar{q}^{*,k}_{1,lj}\right)^{w_k} \\ &= \left(\otimes_{l=1}^m \left(\bar{q}^{*,k}_{1,il}\right)^{w_k}\right) \otimes \left(\otimes_{l=1}^m \left(\bar{q}^{*,k}_{1,lj}\right)^{w_k}\right) = \bar{q}_{1,il} \otimes \bar{q}_{1,lj} \end{aligned}$$

where $w = (w_1, w_2, \ldots, w_m)$ is a weight vector as shown in formula (29). Thus, $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ is completely consistent. Similarly, one can check that $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$ is completely consistent too. □

Definition 16 Let $\tilde{R}^k = (\tilde{r}^k_{ij})_{n\times n}$, $k = 1, 2, \ldots, m$, be any given m IVIMLPRs, let $\bar{Q}_1^{*,k} = (\bar{q}^{*,k}_{1,ij})_{n\times n}$ and $\bar{Q}_2^{*,k} = (\bar{q}^{*,k}_{2,ij})_{n\times n}$ be the associated individual consistent QMILPRs, and let $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ and $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$ be the comprehensive

QMILPRs as shown in Definition 15. Then, the consensus level of $\tilde{R}^k = (\tilde{r}^k_{ij})_{n\times n}$ is defined as:

$$\begin{aligned} &GCI(\tilde{R}^k) \\ &\quad = 1 \\ &\qquad - \frac{1}{4n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left|\log_t(q^{*,k,l}_{1,ij}) - \log_t(q^{l}_{1,ij})\right| + \left|\log_t(q^{*,k,u}_{1,ij}) \ - \log_t(q^{u}_{1,ij})\right| \right. \\ &\qquad \left. + \left|\log_t(q^{*,k,l}_{2,ij}) - \log_t(q^{l}_{2,ij})\right| + \left|\log_t(q^{*,k,u}_{2,ij}) - \log_t(q^{u}_{2,ij})\right| \right) \end{aligned} \tag{30}$$

Following formula (30), we derive $0 \le GCI(\tilde{R}^k) \le 1$ for any IVIMLPR $\tilde{R}^k$.

In the procedure of calculating comprehensive QMILPRs, the weight vector is used. In the setting of GDM, the weights of DMs are usually unknown. Therefore, we first need to determine the weights of DMs for calculating comprehensive QMILPRs. Based on formula (30), we next build an optimization model to determine the weights of DMs.

For all $i, j = 1, 2, \ldots, n$ and all $k = 1, 2, \ldots, m$, formula (30) shows that the smaller the value of the following equation is, the higher the consensus level will be, where

$$\begin{aligned} &\left|\log_t(q^{*,k,l}_{1,ij}) - \log_t(q^{l}_{1,ij})\right| + \left|\log_t(q^{*,k,u}_{1,ij}) - \log_t(q^{u}_{1,ij})\right| \\ &\quad + \left|\log_t(q^{*,k,l}_{2,ij}) - \log_t(q^{l}_{2,ij})\right| + \left|\log_t(q^{*,k,u}_{2,ij}) - \log_t(q^{u}_{2,ij})\right| \end{aligned}$$

Thus, we construct the following optimization model to determine the weights of DMs:

$$\zeta^* = \min \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(c^+_{k,ij} + c^-_{k,ij} + d^+_{k,ij} + d^-_{k,ij} + x^+_{k,ij} + x^-_{k,ij} + y^+_{k,ij} + y^-_{k,ij} \right)$$

$$s.t. \begin{cases} \left. \begin{array}{l} \log_t(q^{*,k,l}_{1,ij}) - \sum_{z=1}^{m} \omega_z \log_t(q^{*,z,l}_{1,ij}) - c^+_{k,ij} + c^-_{k,ij} = 0 \\ \log_t(q^{*,k,u}_{1,ij}) - \sum_{z=1}^{m} \omega_z \log_t(q^{*,z,u}_{1,ij}) - d^+_{k,ij} + d^-_{k,ij} = 0 \\ \log_t(q^{*,k,l}_{2,ij}) - \sum_{z=1}^{m} \omega_z \log_t(q^{*,z,l}_{2,ij}) - x^+_{k,ij} + x^-_{k,ij} = 0 \\ \log_t(q^{*,k,u}_{2,ij}) - \sum_{z=1}^{m} \omega_z \log_t(q^{*,z,u}_{2,ij}) - y^+_{k,ij} + y^-_{k,ij} = 0 \end{array} \right\} \\ i, j = 1, 2, \ldots, n, i < j, k = 1, 2, \ldots, m \\ c^+_{k,ij}, c^-_{k,ij}, d^+_{k,ij}, d^-_{k,ij}, x^+_{k,ij}, x^-_{k,ij}, y^+_{k,ij}, y^-_{k,ij} \ge 0, \\ i, j = 1, 2, \ldots, n, i < j, k = 1, 2, \ldots, m \\ \sum_{z=1}^{m} \omega_z = 1, \omega_z \ge 0, z = 1, 2, \ldots, m \end{cases} \tag{M-6}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ is the weight vector on the DM set.

Let Θ^* be the given threshold of consensus. If the consensus level of the DM e_k's opinion does not satisfy the minimum threshold, namely,$GCI(\tilde{R}^k) < \Theta^*$, then, we need to improve his/her consensus level. Considering that the influences of different judgments are different, their adjustments should be different too. For all $i, j = 1, 2, \ldots, n$ such that $i < j$, let

$$\begin{cases} \left[q_{1,ij}^{'*,k,l}, q_{1,ij}^{'*,k,u}\right] = [(q_{1,ij}^{*,k,l})^{\kappa_{1,ij}^{k,l}}, (q_{1,ij}^{*,k,u})^{\kappa_{1,ij}^{k,u}}] \otimes [(q_{1,ij}^{l})^{1-\kappa_{1,ij}^{k,l}}, (q_{1,ij}^{u})^{1-\kappa_{1,ij}^{k,u}}] \\ \left[q_{2,ij}^{'*,k,l}, q_{2,ij}^{'*,k,u}\right] = [(q_{2,ij}^{*,k,l})^{\kappa_{2,ij}^{k,l}}, (q_{2,ij}^{*,k,u})^{\kappa_{2,ij}^{k,u}}] \otimes [(q_{2,ij}^{l})^{1-\kappa_{2,ij}^{k,l}}, (q_{2,ij}^{u})^{1-\kappa_{2,ij}^{k,u}}] \end{cases} \tag{31}$$

where $\kappa_{1,ij}^{k}, \kappa_{2,ij}^{k} \in (0, 1)$.

Based on formula (31), we have

$$\begin{cases} \log\left(I(q_{1,ij}^{'*,k,l})\right) = \kappa_{1,ij}^{k,l} \log\left(I(q_{1,ij}^{*,k,l})\right) + (1 - \kappa_{1,ij}^{k,l}) \log\left(I(q_{1,ij}^{l})\right) \\ \log\left(I(q_{1,ij}^{'*,k,u})\right) = \kappa_{1,ij}^{k,u} \log\left(I(q_{1,ij}^{*,k,u})\right) + (1 - \kappa_{1,ij}^{k,u}) \log\left(I(q_{1,ij}^{u})\right) \\ \log\left(I(q_{2,ij}^{'*,k,l})\right) = \kappa_{2,ij}^{k,l} \log\left(I(q_{2,ij}^{*,k,l})\right) + (1 - \kappa_{2,ij}^{k,l}) \log\left(I(q_{2,ij}^{l})\right) \\ \log\left(I(q_{2,ij}^{'*,k,u})\right) = \kappa_{2,ij}^{k,u} \log\left(I(q_{2,ij}^{*,k,u})\right) + (1 - \kappa_{2,ij}^{k,u}) \log\left(I(q_{2,ij}^{u})\right) \end{cases} \tag{32}$$

In addition, the adjusted QMILPRs should satisfy the following conditions:

(i) The consensus level of the adjusted QMILPRs should not be smaller than the given consensus threshold;
(ii) The consistency of the adjusted QMILPRs should not change;
(iii) The adjusted QMILPRs should satisfy the conditions of elements in IVIMLPRs;
(iv) The adjustment should be as small as possible for retaining more original information.

Following the above analysis, we construct the following optimization model to improve the consensus level of individual IVIMLPRs:

$$\xi^* = \max \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\kappa_{1,ij}^{k,l} + \kappa_{1,ij}^{k,u} + \kappa_{2,ij}^{k,l} + \kappa_{2,ij}^{k,u}\right)$$

$$s.t. \begin{cases} CA(\tilde{R}^k) \leq 4n(n-1) \times (1-\Theta^*) \log(t) \\ \kappa_{1,ij}^{k,l} \log\Big(I(q_{1,ij}^{*,k,l})\Big) + (1-\kappa_{1,ij}^{k,l}) \log\Big(I(q_{1,ij}^{l})\Big) = \kappa_{1,ih}^{k,l} \log\Big(I(q_{1,ih}^{*,k,l})\Big) + \\ (1-\kappa_{1,ih}^{k,l}) \log\Big(I(q_{1,ih}^{l})\Big) + \kappa_{1,hj}^{k,l} \log\Big(I(q_{1,hj}^{*,k,l})\Big) + (1-\kappa_{1,hj}^{k,l}) \log\Big(I(q_{1,hj}^{l})\Big) \\ \kappa_{1,ij}^{k,u} \log\Big(I(q_{1,ij}^{*,k,u})\Big) + (1-\kappa_{1,ij}^{k,u}) \log\Big(I(q_{1,ij}^{u})\Big) = \kappa_{1,ih}^{k,u} \log\Big(I(q_{1,ih}^{*,k,u})\Big) + \\ (1-\kappa_{1,ih}^{k,u}) \log\big(I(q_{1,ih}^{u})\big) + \kappa_{1,hj}^{k,u} \log\Big(I(q_{1,hj}^{*,k,u})\Big) + (1-\kappa_{1,hj}^{k,u}) \log\Big(I(q_{1,hj}^{u})\Big) \\ \kappa_{2,ij}^{k,l} \log\Big(I(q_{2,ij}^{*,k,l})\Big) + (1-\kappa_{2,ij}^{k,l}) \log\Big(I(q_{2,ij}^{l})\Big) = \kappa_{2,ih}^{k,l} \log\Big(I(q_{2,ih}^{*,k,l})\Big) + \\ (1-\kappa_{2,ih}^{k,l}) \log\Big(I(q_{2,ih}^{l})\Big) + \kappa_{2,hj}^{k,l} \log\Big(I(q_{2,hj}^{*,k,l})\Big) + (1-\kappa_{2,hj}^{k,l}) \log\Big(I(q_{2,hj}^{l})\Big) \\ \kappa_{2,ij}^{k,u} \log\Big(I(q_{2,ij}^{*,k,u})\Big) + (1-\kappa_{2,ij}^{k,u}) \log\Big(I(q_{2,ij}^{u})\Big) = \kappa_{2,ih}^{k,u} \log\Big(I(q_{2,ih}^{*,k,u})\Big) + \\ (1-\kappa_{2,ih}^{k,u}) \log\big(I(q_{2,ih}^{u})\big) + \kappa_{2,hj}^{k,u} \log\Big(I(q_{2,hj}^{*,k,u})\Big) + (1-\kappa_{2,hj}^{k,u}) \log\Big(I(q_{2,hj}^{u})\Big) \\ 0 < \kappa_{1,ij}^{k,l}, \kappa_{1,ij}^{k,u}, \kappa_{2,ij}^{k,l}, \kappa_{2,ij}^{k,u} < 1, i, j = 1, 2, \ldots, n, i < j \\ \gamma_{ij}^1 \rho_{ij}^{1,k} \wedge \gamma_{ij}^1 \rho_{ij}^{2,k} \wedge \gamma_{ij}^1 \rho_{ij}^{3,k} \wedge \gamma_{ij}^1 \rho_{ij}^{4,k} \leq 0, i, j = 1, 2, \ldots, n, i < j \\ \gamma_{ij}^2 \rho_{ij}^{1,k} > 0, \gamma_{ij}^2 \rho_{ij}^{2,k} \wedge \gamma_{ij}^2 \rho_{ij}^{5,k} \wedge \gamma_{ij}^2 \rho_{ij}^{6,k} \leq 0, i, j = 1, 2, \ldots, n, i < j \\ \gamma_{ij}^3 \rho_{ij}^{2,k} > 0, \gamma_{ij}^3 \rho_{ij}^{1,k} \wedge \gamma_{ij}^3 \rho_{ij}^{5,k} \wedge \gamma_{ij}^3 \rho_{ij}^{6,k} \leq 0, i, j = 1, 2, \ldots, n, i < j \\ \gamma_{ij}^4 \rho_{ij}^{1,k} \wedge \gamma_{ij}^4 \rho_{ij}^{2,k,l} > 0, \gamma_{ij}^4 \rho_{ij}^{3,k} \wedge \gamma_{ij}^4 \rho_{ij}^{4,k} \leq 0, i, j = 1, 2, \ldots, n, i < j \\ \gamma_{ij}^1, \gamma_{ij}^2, \gamma_{ij}^3, \gamma_{ij}^4 = 0 \vee 1, i, j = 1, 2, \ldots n, i < j \\ \gamma_{ij}^1 + \gamma_{ij}^2 + \gamma_{ij}^3 + \gamma_{ij}^4 = 1, i, j = 1, 2, \ldots, n, i < j \end{cases} \tag{M-7}$$

where

$$\begin{cases} \rho_{ij}^{1,k} = \kappa_{1,ij}^{k,l} \log\Big(I(q_{1,ij}^{*,k,l})\Big) + (1-\kappa_{1,ij}^{k,l}) \log\Big(I(q_{1,ij}^{l})\Big) - \kappa_{1,ij}^{k,u} \log\Big(I(q_{1,ij}^{*,k,u})\Big) - (1-\kappa_{1,ij}^{k,u}) \log\Big(I(q_{1,ij}^{u})\Big) \\ \rho_{ij}^{2,k} = \kappa_{2,ij}^{k,l} \log\Big(I(q_{2,ij}^{*,k,l})\Big) + (1-\kappa_{2,ij}^{k,l}) \log\Big(I(q_{2,ij}^{l})\Big) - \kappa_{2,ij}^{k,u} \log\Big(I(q_{2,ij}^{*,k,u})\Big) - (1-\kappa_{2,ij}^{k,u}) \log\Big(I(q_{2,ij}^{u})\Big) \\ \rho_{ij}^{3,k} = \kappa_{1,ij}^{k,l} \log\Big(I(q_{1,ij}^{*,k,l})\Big) + (1-\kappa_{1,ij}^{k,l}) \log\Big(I(q_{1,ij}^{l})\Big) - \kappa_{2,ij}^{k,l} \log\Big(I(q_{2,ij}^{*,k,l})\Big) - (1-\kappa_{2,ij}^{k,l}) \log\Big(I(q_{2,ij}^{l})\Big) \\ \rho_{ij}^{4,k} = \kappa_{1,ij}^{k,u} \log\Big(I(q_{1,ij}^{*,k,u})\Big) + (1-\kappa_{1,ij}^{k,u}) \log\Big(I(q_{1,ij}^{u})\Big) - \kappa_{2,ij}^{k,u} \log\Big(I(q_{2,ij}^{*,k,u})\Big) - (1-\kappa_{2,ij}^{k,u}) \log\Big(I(q_{2,ij}^{u})\Big) \\ \rho_{ij}^{5,k} = \kappa_{1,ij}^{k,u} \log\Big(I(q_{1,ij}^{*,k,u})\Big) + (1-\kappa_{1,ij}^{k,u}) \log\Big(I(q_{1,ij}^{u})\Big) - \kappa_{2,ij}^{k,l} \log\Big(I(q_{2,ij}^{*,k,l})\Big) - (1-\kappa_{2,ij}^{k,l}) \log\Big(I(q_{2,ij}^{l})\Big) \\ \rho_{ij}^{6,k} = \kappa_{1,ij}^{k,l} \log\Big(I(q_{1,ij}^{*,k,l})\Big) + (1-\kappa_{1,ij}^{k,l}) \log\Big(I(q_{1,ij}^{l})\Big) - \kappa_{2,ij}^{k,u} \log\Big(I(q_{2,ij}^{*,k,u})\Big) - (1-\kappa_{2,ij}^{k,u}) \log\Big(I(q_{2,ij}^{u})\Big) \end{cases}$$

and

$$\begin{aligned} CA(\tilde{R}^k) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Bigg(& \Big|\kappa_{1,ij}^{k,l} \log\Big(I(q_{1,ij}^{*,k,l})\Big) + (1-\kappa_{1,ij}^{k,l}) \log\Big(I(q_{1,ij}^{l})\Big) \\ & - \Big|\omega_k \Big(\kappa_{1,ij}^{k,l} \log\Big(I(q_{1,ij}^{*,k,l})\Big) + (1-\kappa_{1,ij}^{k,l}) \log\Big(I(q_{1,ij}^{l})\Big)\Big) \\ & - \Big(\sum_{z=1, z\neq k}^{m} \omega_z \log(q_{1,ij}^{*,z,l})\Big)\Big| \end{aligned}$$

$$
\begin{aligned}
&+\left|\kappa_{1,ij}^{k,u}\log\left(I(q_{1,ij}^{*,k,u})\right)+(1-\kappa_{1,ij}^{k,u})\log\left(I(q_{1,ij}^{u})\right)\right.\\
&-\omega_k\left(\kappa_{1,ij}^{k,u}\log\left(I(q_{1,ij}^{*,k,u})\right)+(1-\kappa_{1,ij}^{k,u})\log\left(I(q_{1,ij}^{u})\right)\right)\\
&\left.-\left(\sum\nolimits_{z=1,z\neq k}^{m}\omega_z\times\left(\sum\nolimits_{z=1,z\neq k}^{m}\omega_z\log(q_{1,ij}^{*,z,l})\right)\right)\right|\\
&+\left|\kappa_{1,ij}^{k,u}\log\left(I(q_{1,ij}^{*,k,u})\right)+(1-\kappa_{1,ij}^{k,u})\log\left(I(q_{1,ij}^{u})\right)\right.\\
&-\omega_k\left(\kappa_{1,ij}^{k,u}\log\left(I(q_{1,ij}^{*,k,u})\right)+(1-\kappa_{1,ij}^{k,u})\log\left(I(q_{1,ij}^{u})\right)\right)\\
&\left.-\left(\sum\nolimits_{z=1,z\neq k}^{m}\omega_z\times\log(q_{1,ij}^{*,z,u})\right)\right|\\
&+\left|\kappa_{2,ij}^{k,l}\log\left(I(q_{2,ij}^{*,k,l})\right)+(1-\kappa_{2,ij}^{k,l})\log\left(I(q_{2,ij}^{l})\right)-\omega_k\times\left(\kappa_{2,ij}^{k,l}\log\left(I(q_{2,ij}^{*,k,l})\right)\right.\right.\\
&\left.\left.+(1-\kappa_{2,ij}^{k,l})\log\left(I(q_{2,ij}^{l})\right)\right)-\left(\sum\nolimits_{z=1,z\neq k}^{m}\omega_z\log(q_{2,ij}^{*,z,l})\right)\right|\\
&+\left|\kappa_{2,ij}^{k,u}\log\left(I(q_{2,ij}^{*,k,u})\right)+(1-\kappa_{2,ij}^{k,u})\log\left(I(q_{2,ij}^{u})\right)-\omega_k\left(\kappa_{2,ij}^{k,u}\log\left(I(q_{2,ij}^{*,k,u})\right)\right.\right.\\
&\left.\left.+(1-\kappa_{2,ij}^{k,u})\log\left(I(q_{2,ij}^{u})\right)\right)-\left(\sum\nolimits_{z=1,z\neq k}^{m}\omega_z\log(q_{2,ij}^{*,z,u})\right)\right|
\end{aligned}
$$

6.2 A consistency and Consensus Based-Algorithm for GDM

Based on the consistency and consensus analysis, this subsection offers an algorithm for GDM with IVIMLPRs

Step 1 Let $\tilde{R}^k=(\tilde{r}_{ij}^k)_{n\times n}$ be the individual IVIMLPR offered by the DM e_k, where $k=1,2,\ldots,m$. If all of them are complete, go to Step 2. Otherwise, model (M-2) is used to determine unknown linguistic variables in each incompletely individual IVIMLPR, which is still denoted by $\tilde{R}^k=(\tilde{r}_{ij}^k)_{n\times n}$, $k=1,2,\ldots,m$;

Step 2 For each completely individual IVIMLPR $\tilde{R}^k=(\tilde{r}_{ij}^k)_{n\times n}$, $k=1,2,\ldots,m$, model (M-1) is adopted to judge its consistency. If the value of objective function is zero, then the associated individual IVIMLPR is consistent, and skip to Step 4. Otherwise, go to Step 3;

Step 3 For each inconsistent individual IVIMLPR $\tilde{R}^k=(\tilde{r}_{ij}^k)_{n\times n}$, $k=1,2,\ldots,m$, models (M-3) and (M-5) are applied to improve the consistency. The associated individual QMILPRs are denoted as: $\bar{Q}_1^{*,k}=(\bar{q}_{1,ij}^{*,k})_{n\times n}$ and $\bar{Q}_2^{*,k}=(\bar{q}_{2,ij}^{*,k})_{n\times n}$, $k=1,2,\ldots,m$;

Step 4 Based on individual consistent QMILPRs, model (M-6) is adopted to determine the weights of the DMs, denoted as $\omega=(\omega_1,\omega_2,\ldots,\omega_m)$;

Step 5 Formula (29) is used to calculate the comprehensively consistent QMILPRs $\bar{Q}_1^*=(\bar{q}_{1,ij}^*)_{n\times n}$ and $\bar{Q}_2^*=(\bar{q}_{2,ij}^*)_{n\times n}$. If the conditions of elements in the associated IVIMLPRs is not true, then model (M-5) is applied to adjust the consistency of $\bar{Q}_1^*=(\bar{q}_{1,ij}^*)_{n\times n}$ and $\bar{Q}_2^*=(\bar{q}_{2,ij}^*)_{n\times n}$ again. If there is no fear of confusion, the adjusted consistent QMILPRs are still denoted as $\bar{Q}_1^*=(\bar{q}_{1,ij}^*)_{n\times n}$ and $\bar{Q}_2^*=(\bar{q}_{2,ij}^*)_{n\times n}$;

Step 6 Let Θ^* be the given threshold of consensus. Formula (31) is utilized to adjust the consensus of each individual IVIMLPR $\tilde{R}^k = (\tilde{r}_{ij}^k)_{n\times n}$. If $\min_{1\le l\le m} GCI(\tilde{R}^l) \ge \Theta^*$, skip to Step 8. Otherwise, go to the next step;

Step 7 Let $GCI(\tilde{R}^k) = \min_{1\le l\le m} GCI(\tilde{R}^l) < \Theta^*$. Model (M-7) is used to improve the consensus level of QMILPRs $\bar{Q}_1^{*,k} = (\bar{q}_{1,ij}^{*,k})_{n\times n}$ and $\bar{Q}_2^{*,k} = (\bar{q}_{2,ij}^{*,k})_{n\times n}$. If there is no fear of confusion, we still use $\bar{Q}_1^{*,k} = (\bar{q}_{1,ij}^{*,k})_{n\times n}$ and $\bar{Q}_2^{*,k} = (\bar{q}_{2,ij}^{*,k})_{n\times n}$ to denote them and return to Step 5;

Step 8 Based on the comprehensively consistent QMILPRs $\bar{Q}_1^* = (\bar{q}_{1,ij}^*)_{n\times n}$ and $\bar{Q}_2^* = (\bar{q}_{2,ij}^*)_{n\times n}$, the comprehensively consistent IVIMLPR $\tilde{R}^* = (\tilde{r}_{ij}^*)_{n\times n} = \left([s^*_{\mu_{ij}^-}, s^*_{\mu_{ij}^+}], [s^*_{v_{ij}^-}, s^*_{v_{ij}^+}]\right)_{n\times n}$ can be derived. The interval-valued intuitionistic multiplicative linguistic priority weights are

$$\tilde{w}_i = \left(\left[\sqrt[n]{\otimes_{j=1}^n s^*_{\mu_{ij}^-}}, \sqrt[n]{\otimes_{j=1}^n s^*_{\mu_{ij}^+}}\right], \left[\sqrt[n]{\otimes_{j=1}^n s^*_{v_{ij}^-}}, \sqrt[n]{\otimes_{j=1}^n s^*_{v_{ij}^+}}\right]\right),$$
$$i = 1, 2, \ldots, n; \tag{33}$$

Step 9 Formulae (6) and (7) are used to calculate the score and accuracy values of $\tilde{w}_i$, $i = 1, 2,\ldots, n$, by which the ranking of objects $x_1, x_2, \ldots, x_n$ can be derived.

To see the procedure of the above algorithm intuitively, please see Fig. 2.

7 A Case Study

With the development of human social and economic activities and information technology, the production and sales enterprises are more and more dependent on the technical level and economic scale of their partners. The status and role of electing suitable partners become more and more important for making more benefits and gaining competitive advantages. Therefore, enterprises pay more and more attentions to the choice of partners. Because there is usually more than one evaluating factor, and the situation where one potential partner is superior to other potential partners for all considered factors seldom exists, it is difficult for the DMs to offer their quantitative judgments. Considering that linguistic fuzzy variables are powerful to express the subjective vagueness of human and more convenient for DMs to make judgments, IVIMLVs are a good choice as they can denote the asymmetrically uncertain preferred and non-preferred qualitative judgments. There is a coal thermal power enterprise to select coal suppliers. After the initial investigations, four coal companies are selected as potential partners. To select the most suitable partners, four DMs are invited to evaluate them based on having information and their expertise. When the DMs do not offer some comparisons, unknown linguistic information is permitted. Let $S = \{s_{1/5}$: very bad; $s_{1/4}$: very poor; $s_{1/3}$: poor; $s_{1/2}$: slightly worse; s_1: not bad; s_2: slightly

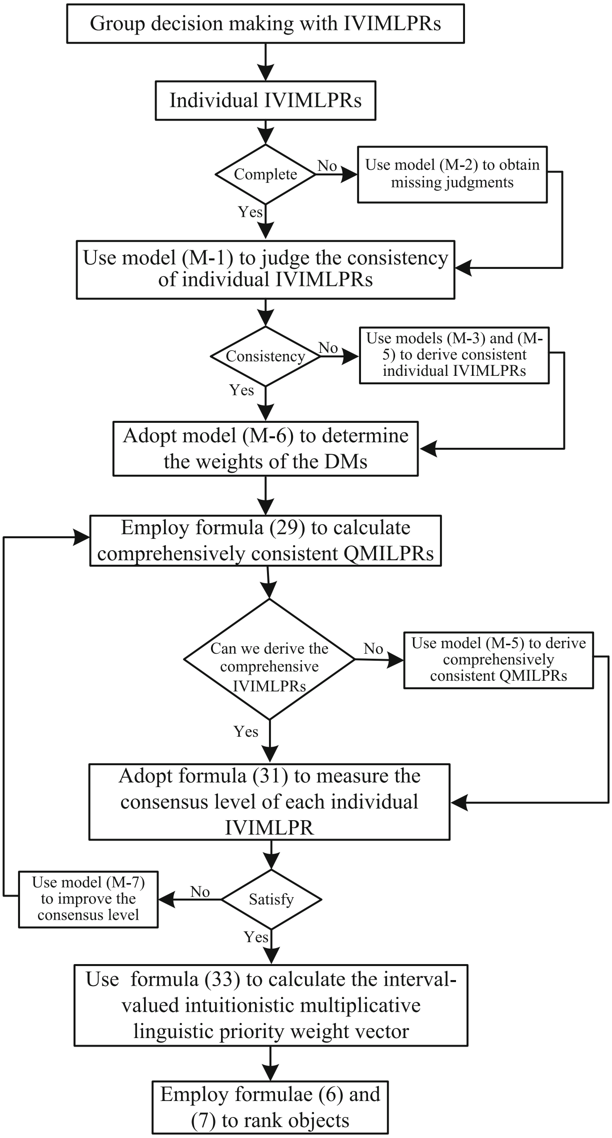

Fig. 2 The framework of the above algorithm

better; s_3: good; s_4: very good; s_5: excellent} be the given DALTS. Assume that the individual IVIMLPRs are provided as follows:

$$\tilde{R}^1 = \begin{pmatrix} ([s_1, s_1], [s_1, s_1]) & ([s_1, s_2], [s_{1/3}, s_{1/2}]) & ([s_{1/2}, s_1], [s_1, s_2]) & ([s_{1/4}, s_{1/2}], [s_1, s_2]) \\ ([s_{1/3}, s_{1/2}], [s_1, s_2]) & ([s_1, s_1], [s_1, s_1]) & \left([s_{\mu_{23}^-}, s_{\mu_{23}^+}], [s_{\upsilon_{23}^-}, s_{\upsilon_{23}^+}]\right) & \left([s_{\mu_{24}^-}, s_{\mu_{24}^+}], [s_{\upsilon_{24}^-}, s_{\upsilon_{24}^+}]\right) \\ ([s_1, s_2], [s_{1/2}, s_1]) & \left([s_{\mu_{32}^-}, s_{\mu_{32}^+}], [s_{\upsilon_{32}^-}, s_{\upsilon_{32}^+}]\right) & ([s_1, s_1], [s_1, s_1]) & ([s_1, s_3], [s_{1/4}, s_1]) \\ ([s_1, s_2], [s_{1/4}, s_{1/2}]) & \left([s_{\mu_{42}^-}, s_{\mu_{42}^+}], [s_{\upsilon_{42}^-}, s_{\upsilon_{42}^+}]\right) & ([s_{1/4}, s_1], [s_1, s_3]) & ([s_1, s_1], [s_1, s_1]) \end{pmatrix}$$

$$\tilde{R}^2 = \begin{pmatrix} ([s_1, s_1], [s_1, s_1]) & \left([s_{\mu_{12}^-}, s_{\mu_{12}^+}], [s_{v_{12}^-}, s_{v_{12}^+}]\right) & \left([s_{\mu_{13}^-}, s_{\mu_{13}^+}], [s_{v_{13}^-}, s_{v_{13}^+}]\right) & ([s_{1/3}, s_1], [s_{1/2}, s_2]) \\ \left([s_{\mu_{21}^-}, s_{\mu_{21}^+}], [s_{v_{21}^-}, s_{v_{21}^+}]\right) & ([s_1, s_1], [s_1, s_1]) & ([s_2, s_3], [s_{1/4}, s_{1/2}]) & ([s_{1/3}, s_{1/2}], [s_1, s_2]) \\ \left([s_{\mu_{31}^-}, s_{\mu_{31}^+}], [s_{v_{31}^-}, s_{v_{31}^+}]\right) & ([s_{1/4}, s_{1/2}], [s_2, s_3]) & ([s_1, s_1], [s_1, s_1]) & ([s_1, s_2], [s_{1/3}, s_1]) \\ ([s_{1/2}, s_2], [s_{1/3}, s_1]) & ([s_1, s_2], [s_{1/3}, s_{1/2}]) & ([s_{1/3}, s_1], [s_1, s_2]) & ([s_1, s_1], [s_1, s_1]) \end{pmatrix}$$

$$\tilde{R}^3 = \begin{pmatrix} ([s_1, s_1], [s_1, s_1]) & ([s_2, s_3], [s_{1/4}, s_{1/2}]) & \left([s_{\mu_{13}^-}, s_{\mu_{13}^+}], [s_2, s_4]\right) & \left([s_{\mu_{14}^-}, s_{\mu_{14}^+}], [s_{v_{14}^-}, s_{v_{14}^+}]\right) \\ ([s_{1/4}, s_{1/2}], [s_2, s_3]) & ([s_1, s_1], [s_1, s_1]) & ([s_2, s_3], [s_{1/4}, s_{1/2}]) & ([s_{1/2}, s_1], [s_1, s_2]) \\ \left([s_2, s_4], [s_{v_{13}^-}, s_{v_{13}^+}]\right) & ([s_{1/4}, s_{1/2}], [s_2, s_3]) & ([s_1, s_1], [s_1, s_1]) & \left([s_{\mu_{34}^-}, s_{\mu_{34}^+}], [s_{v_{34}^-}, s_{v_{34}^+}]\right) \\ \left([s_{\mu_{41}^-}, s_{\mu_{41}^+}], [s_{v_{41}^-}, s_{v_{41}^+}]\right) & ([s_{1/2}, s_1], [s_1, s_2]) & \left([s_{\mu_{43}^-}, s_{\mu_{43}^+}], [s_{v_{43}^-}, s_{v_{43}^+}]\right) & ([s_1, s_1], [s_1, s_1]) \end{pmatrix}$$

$$\tilde{R}^4 = \begin{pmatrix} ([s_1, s_1], [s_1, s_1]) & \left([s_{\mu_{12}^-}, s_{\mu_{12}^+}], [s_{1/3}, s_1]\right) & \left([s_1, s_2], [s_{v_{13}^-}, s_{v_{13}^+}]\right) & ([s_{1/3}, s_{1/2}], [s_1, s_2]) \\ \left([s_{1/3}, s_1], [s_{v_{21}^-}, s_{v_{21}^+}]\right) & ([s_1, s_1], [s_1, s_1]) & ([s_2, s_4], [s_{1/4}, s_{1/3}]) & \left([s_{\mu_{24}^-}, s_{\mu_{24}^+}], [s_{v_{24}^-}, s_{v_{24}^+}]\right) \\ \left([s_{\mu_{31}^-}, s_{\mu_{31}^+}], [s_1, s_2]\right) & ([s_{1/4}, s_{1/3}], [s_2, s_4]) & ([s_1, s_1], [s_1, s_1]) & ([s_2, s_3], [s_{1/4}, s_{1/3}]) \\ ([s_1, s_2], [s_{1/3}, s_{1/2}]) & \left([s_{\mu_{42}^-}, s_{\mu_{42}^+}], [s_{v_{42}^-}, s_{v_{42}^+}]\right) & ([s_{1/4}, s_{1/3}], [s_2, s_3]) & ([s_1, s_1], [s_1, s_1]) \end{pmatrix}$$

Next, we apply the algorithm listed in Subsection 6.2 to rank these four coal companies following the above individual IVIMLPRs:

Step 1: For each individual incomplete IVIMLPR, the unknown linguistic variables based on model (M-2) are derived as follows:

$$\begin{cases} \tilde{r}_{23}^1 = ([s_{0.25}, s_{0.25}], [s_{3.17}, s_{3.17}]) \\ \tilde{r}_{24}^1 = ([s_{0.25}, s_{0.4}], [s_2, s_{3.56}]) \end{cases},$$

$$\begin{cases} \tilde{r}_{12}^2 = ([s_{0.55}, s_{0.87}], [s_{1.14}, s_{1.59}]) \\ \tilde{r}_{13}^2 = ([s_{0.61}, s_{1.14}], [s_{0.66}, s_{1.26}]) \end{cases},$$

$$\begin{cases} \left[s^3_{\mu_{13}^-}, s^3_{\mu_{13}^+}\right] = [s_{0.25}, s_{0.5}] \\ \tilde{r}_{14}^3 = ([s_{0.4}, s_{1.14}], [s_{0.79}, s_{2.52}]) \\ \tilde{r}_{34}^3 = ([s_{0.63}, s_{0.79}], [s_{1.14}, s_{1.59}]) \end{cases}, \quad \begin{cases} \left[s^4_{\mu_{12}^-}, s^4_{\mu_{12}^+}\right] = [s_{0.2}, s_{0.22}] \\ [s^4_{v_{13}^-}, s^4_{v_{13}^+}] = [s_{0.5}, s_1] \\ \tilde{r}_{24}^4 = ([s_{1.15}, s_{2.31}], [s_{0.2}, s_{0.21}]) \end{cases}$$

Step 2: Model (M-1) is used to judge the consistency of each individual complete IVIMLPR, we have

$$f_1^* = 6.3561, \quad f_2^* = 16.2957, \quad f_3^* = 7.9552, \quad f_3^* = 21.2655.$$

Step 3: For each individual complete IVIMLPR, the optimal 0–1 indictor variables based on model (M-3) are obtained as:

$$\begin{cases} \alpha_{12} = \alpha_{14} = 0, \alpha_{13} = \alpha_{24} = \alpha_{34} = \alpha_{23} = 1 \\ \beta_{12} = \beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = \beta_{34} = 1 \end{cases}, \quad \begin{cases} \alpha_{12} = \alpha_{14} = \alpha_{13} = \alpha_{24} = \alpha_{34} = \alpha_{23} = 1 \\ \beta_{12} = \beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = \beta_{34} = 1 \end{cases}$$

$$\begin{cases} \alpha_{12} = \alpha_{14} = \alpha_{13} = \alpha_{24} = \alpha_{34} = \alpha_{23} = 1 \\ \beta_{12} = \beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = \beta_{34} = 1 \end{cases}, \quad \begin{cases} \alpha_{12} = 0, \alpha_{13} = \alpha_{14} = \alpha_{24} = \alpha_{34} = \alpha_{23} = 1 \\ \beta_{12} = \beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = \beta_{34} = 1 \end{cases}$$

Using these 0–1 indicator variables, the individual QMILPRs can be derived. Taking the individual complete IVIMLPR $\tilde{R}^1$ for example, the associated individual QMILPRs are

$$\bar{Q}_1^1 = \begin{pmatrix} [s_1, s_1] & [s_2, s_1] & [s_{0.5}, s_{0.5}] & [s_{0.5}, s_{0.25}] \\ [s_{0.5}, s_1] & [s_1, s_1] & [s_{0.25}, s_{0.31}] & [s_{0.25}, s_{0.28}] \\ [s_2, s_2] & [s_4, s_{3.17}] & [s_1, s_1] & [s_1, s_1] \\ [s_2, s_4] & [s_4, s_{3.56}] & [s_1, s_1] & [s_1, s_1] \end{pmatrix},$$

$$\bar{Q}_2^1 = \begin{pmatrix} [s_1, s_1] & [s_2, s_3] & [s_1, s_1] & [s_{0.5}, s_1] \\ [s_{0.5}, s_{0.33}] & [s_1, s_1] & [s_{0.25}, s_{0.31}] & [s_{0.4}, s_{0.5}] \\ [s_1, s_1] & [s_4, s_{3.17}] & [s_1, s_1] & [s_3, s_4] \\ [s_2, s_1] & [s_{2.5}, s_2] & [s_{0.33}, s_{0.25}] & [s_1, s_1] \end{pmatrix}$$

With respect to per individual QMILPR, the individual consistent QMILPRs based on model (M-5) are obtained as:

$$\bar{Q}_1^{1,*} = \begin{pmatrix} [s_1, s_1] & [s_2, s_1] & [s_{0.5}, s_{0.31}] & [s_{0.5}, s_{0.28}] \\ [s_{0.5}, s_1] & [s_1, s_1] & [s_{0.25}, s_{0.31}] & [s_{0.25}, s_{0.28}] \\ [s_2, s_{3.23}] & [s_4, s_{3.23}] & [s_1, s_1] & [s_1, s_{0.9}] \\ [s_2, s_{3.57}] & [s_4, s_{3.57}] & [s_1, s_{1.11}] & [s_1, s_1] \end{pmatrix},$$

$$\bar{Q}_2^{1,*} = \begin{pmatrix} [s_1, s_1] & [s_2, s_3] & [s_{0.5}, s_{0.93}] & [s_{0.8}, s_{1.5}] \\ [s_{0.5}, s_{0.33}] & [s_1, s_1] & [s_{0.25}, s_{0.31}] & [s_{0.4}, s_{0.5}] \\ [s_2, s_{1.08}] & [s_4, s_{3.23}] & [s_1, s_1] & [s_{1.6}, s_{1.61}] \\ [s_{1.25}, s_{0.67}] & [s_{2.5}, s_2] & [s_{0.62}, s_{0.62}] & [s_1, s_1] \end{pmatrix}$$

$$\bar{Q}_1^{2,*} = \begin{pmatrix} [s_1, s_1] & [s_{0.55}, s_{0.88}] & [s_{0.61}, s_{0.79}] & [s_{0.33}, s_{0.5}] \\ [s_{1.82}, s_{1.14}] & [s_1, s_1] & [s_{1.11}, s_{0.9}] & [s_{0.6}, s_{0.57}] \\ [s_{1.64}, s_{1.26}] & [s_{0.9}, s_{1.11}] & [s_1, s_1] & [s_{0.54}, s_{0.63}] \\ [s_{3.03}, s_2] & [s_{1.67}, s_{1.76}] & [s_{1.85}, s_{1.58}] & [s_1, s_1] \end{pmatrix},$$

$$\bar{Q}_2^{2,*} = \begin{pmatrix} [s_1, s_1] & [s_{0.87}, s_{0.88}] & [s_{1.14}, s_{1.52}] & [s_1, s_2] \\ [s_{1.15}, s_{1.14}] & [s_1, s_1] & [s_{1.31}, s_{1.73}] & [s_{1.15}, s_{2.27}] \\ [s_{0.88}, s_{0.66}] & [s_{0.76}, s_{0.58}] & [s_1, s_1] & [s_{0.88}, s_{1.32}] \\ [s_1, s_{0.5}] & [s_{0.87}, s_{0.44}] & [s_{1.14}, s_{0.76}] & [s_1, s_1] \end{pmatrix}$$

$$\bar{Q}_1^{3,*} = \begin{pmatrix} [s_1, s_1] & [s_{0.8}, s_{0.8}] & [s_{0.63}, s_{0.63}] & [s_{0.4}, s_{0.4}] \\ [s_{1.25}, s_{1.25}] & [s_1, s_1] & [s_{0.79}, s_{0.79}] & [s_{0.5}, s_{0.5}] \\ [s_{1.58}, s_{1.58}] & [s_{1.26}, s_{1.26}] & [s_1, s_1] & [s_{0.63}, s_{0.63}] \\ [s_{2.5}, s_{2.52}] & [s_2, s_2] & [s_{1.59}, s_{1.59}] & [s_1, s_1] \end{pmatrix},$$

$$\bar{Q}_2^{3,*} = \begin{pmatrix} [s_1, s_1] & [s_{1.14}, s_{1.27}] & [s_{1.44}, s_{1.44}] & [s_{1.14}, s_{1.27}] \\ [s_{0.88}, s_{0.79}] & [s_1, s_1] & [s_{1.27}, s_{1.14}] & [s_1, s_1] \\ [s_{0.69}, s_{0.69}] & [s_{0.79}, s_{0.88}] & [s_1, s_1] & [s_{0.79}, s_{0.88}] \\ [s_{0.88}, s_{0.79}] & [s_1, s_1] & [s_{1.27}, s_{1.14}] & [s_1, s_1] \end{pmatrix}$$

$$\bar{Q}_1^{4,*} = \begin{pmatrix} [s_1, s_1] & [s_{0.22}, s_{0.5}] & [s_{0.44}, s_1] & [s_{0.33}, s_{2.38}] \\ [s_{4.55}, s_2] & [s_1, s_1] & [s_2, s_2] & [s_{1.5}, s_{4.76}] \\ [s_{2.27}, s_1] & [s_{0.5}, s_{0.5}] & [s_1, s_1] & [s_{0.75}, s_{2.38}] \\ [s_{3.03}, s_{0.42}] & [s_{0.67}, s_{0.21}] & [s_{1.33}, s_{0.42}] & [s_1, s_1] \end{pmatrix},$$

$$\bar{Q}_2^{4,*} = \begin{pmatrix} [s_1, s_1] & [s_{0.22}, s_1] & [s_{0.88}, s_2] & [s_{0.66}, s_5] \\ [s_{4.55}, s_1] & [s_1, s_1] & [s_4, s_2] & [s_3, s_5] \\ [s_{1.14}, s_{0.5}] & [s_{0.25}, s_{0.5}] & [s_1, s_1] & [s_{0.75}, s_{2.5}] \\ [s_{1.52}, s_{0.2}] & [s_{0.33}, s_{0.2}] & [s_{1.33}, s_{0.4}] & [s_1, s_1] \end{pmatrix}$$

Step 4: Based on individual consistent QMILPRs, the weight vector on the DM set based on model (M-6) is $\omega = (0.0447, 0.7283, 0.1841, 0.0429)$.

Step 5: The comprehensively consistent QMILPRs using formula (29) are derived as:

$$\bar{Q}_1^* = \begin{pmatrix} [s_1, s_1] & [s_{0.6}, s_{0.85}] & [s_{0.6}, s_{0.73}] & [s_{0.35}, s_{0.5}] \\ [s_{1.67}, s_{1.18}] & [s_1, s_1] & [s_1, s_{0.87}] & [s_{0.58}, s_{0.59}] \\ [s_{1.67}, s_{1.36}] & [s_1, s_{1.15}] & [s_1, s_1] & [s_{0.58}, s_{0.68}] \\ [s_{2.87}, s_2] & [s_{1.73}, s_{1.7}] & [s_{1.73}, s_{1.47}] & [s_1, s_1] \end{pmatrix},$$

$$\bar{Q}_2^* = \begin{pmatrix} [s_1, s_1] & [s_{0.89}, s_1] & [s_{1.13}, s_{1.49}] & [s_1, s_{1.89}] \\ [s_{1.12}, s_1] & [s_1, s_1] & [s_{1.27}, s_{1.49}] & [s_{1.11}, s_{1.89}] \\ [s_{0.88}, s_{0.67}] & [s_{0.79}, s_{0.67}] & [s_1, s_1] & [s_{0.88}, s_{1.27}] \\ [s_1, s_{0.53}] & [s_{0.9}, s_{0.53}] & [s_{1.14}, s_{0.79}] & [s_1, s_1] \end{pmatrix}$$

Step 6: Let $\Theta^* = 0.9$. Following formula (31), we have

$$GCI(\tilde{R}^1) = 0.7644,\ GCI(\tilde{R}^2) = 0.9831,\ GCI(\tilde{R}^3) = 0.942,\ GCI(\tilde{R}^4) = 0.7744$$

Step 7: Because $GCI(\tilde{R}^1) = \min_{1 \le l \le 4} GCI(\tilde{R}^l) = 0.7644 < 0.9$. Model (M-7) is adopted to improve the consensus level of individual consistent QMILPRs $\bar{Q}_1^{*,1}$ and $\bar{Q}_2^{*,1}$. Following formula (31), we have

$$GCI(\tilde{R}^1) = 0.9052,\ GCI(\tilde{R}^2) = 0.9866,\ GCI(\tilde{R}^3) = 0.9365,\ GCI(\tilde{R}^4) = 0.7791$$

Because $GCI(\tilde{R}^4) = \min_{1 \le l \le 4} GCI(\tilde{R}^l) = 0.7791 < 0.9$, we need to use model (M-7) to improve the consensus level of individual consistent QMILPRs $\bar{Q}_1^{*,4}$ and $\bar{Q}_2^{*,4}$, where the adjusted individual consistent QMILPRs are

$$\bar{Q}_1^{1,*} = \begin{pmatrix} [s_1, s_1] & [s_{0.86}, s_1] & [s_{0.5}, s_{0.44}] & [s_{0.5}, s_{0.4}] \\ [s_{1.16}, s_1] & [s_1, s_1] & [s_{0.58}, s_{0.44}] & [s_{0.58}, s_{0.4}] \\ [s_2, s_{2.27}] & [s_{1.72}, s_{2.27}] & [s_1, s_1] & [s_1, s_{0.9}] \\ [s_2, s_{2.52}] & [s_{1.72}, s_{2.52}] & [s_1, s_{1.11}] & [s_1, s_1] \end{pmatrix},$$

$$\bar{Q}_2^{1,*} = \begin{pmatrix} [s_1, s_1] & [s_{0.89}, s_1] & [s_{0.89}, s_{0.93}] & [s_{0.8}, s_{1.5}] \\ [s_{1.12}, s_1] & [s_1, s_1] & [s_1, s_{0.93}] & [s_{0.9}, s_{1.5}] \\ [s_{1.12}, s_{1.07}] & [s_1, s_{1.07}] & [s_1, s_1] & [s_{0.9}, s_{1.61}] \\ [s_{1.25}, s_{0.67}] & [s_{1.11}, s_{0.67}] & [s_{1.11}, s_{0.62}] & [s_1, s_1] \end{pmatrix}$$

$$\bar{Q}_1^{4,*} = \begin{pmatrix} [s_1, s_1] & [s_{0.22}, s_{0.74}] & [s_{0.44}, s_1] & [s_{0.33}, s_{0.68}] \\ [s_{4.55}, s_{1.35}] & [s_1, s_1] & [s_2, s_{1.35}] & [s_{1.5}, s_{0.92}] \\ [s_{2.27}, s_1] & [s_{0.5}, s_{0.74}] & [s_1, s_1] & [s_{0.75}, s_{0.68}] \\ [s_{3.03}, s_{1.47}] & [s_{0.67}, s_{1.09}] & [s_{1.33}, s_{1.47}] & [s_1, s_1] \end{pmatrix},$$

$$\bar{Q}_2^{4,*} = \begin{pmatrix} [s_1, s_1] & [s_{0.46}, s_1] & [s_{0.88}, s_2] & [s_{0.66}, s_{2.54}] \\ [s_{2.19}, s_1] & [s_1, s_1] & [s_{1.93}, s_2] & [s_{1.44}, s_{2.54}] \\ [s_{1.14}, s_{0.5}] & [s_{0.52}, s_{0.5}] & [s_1, s_1] & [s_{0.75}, s_{1.27}] \\ [s_{1.52}, s_{0.39}] & [s_{0.69}, s_{0.39}] & [s_{1.33}, s_{0.79}] & [s_1, s_1] \end{pmatrix}$$

Furthermore, the corresponding comprehensive consistent QMILPRs are

$$\bar{Q}_1^* = \begin{pmatrix} [s_1, s_1] & [s_{0586}, s_{0.86}] & [s_{0.6}, s_{0.75}] & [s_{0.35}, s_{0.48}] \\ [s_{1.73}, s_{1.16}] & [s_1, s_1] & [s_{1.04}, s_{0.87}] & [s_{0.6}, s_{0.56}] \\ [s_{1.67}, s_{1.34}] & [s_{0.96}, s_{1.15}] & [s_1, s_1] & [s_{0.58}, s_{0.64}] \\ [s_{2.87}, s_{2.08}] & [s_{1.66}, s_{1.79}] & [s_{1.73}, s_{1.55}] & [s_1, s_1] \end{pmatrix},$$

$$\bar{Q}_2^* = \begin{pmatrix} [s_1, s_1] & [s_{0.89}, s_{0.95}] & [s_{1.16}, s_{1.49}] & [s_1, s_{1.83}] \\ [s_{1.12}, s_{1.05}] & [s_1, s_1] & [s_{1.31}, s_{1.57}] & [s_{1.12}, s_{1.93}] \\ [s_{0.86}, s_{0.67}] & [s_{0.76}, s_{0.64}] & [s_1, s_1] & [s_{0.86}, s_{1.23}] \\ [s_1, s_{0.55}] & [s_{0.89}, s_{0.52}] & [s_{1.17}, s_{0.81}] & [s_1, s_1] \end{pmatrix}$$

Following formula (31), we have $GCI(\tilde{R}^1) = 0.908$, $GCI(\tilde{R}^2) = 0.9864$, $GCI(\tilde{R}^3) = 0.9414$, $GCI(\tilde{R}^4) = 0.9$.

Step 8: Based on the comprehensively consistent QMILPRs $\bar{Q}_1^* = (\bar{q}_{1,ij}^*)_{n\times n}$ and $\bar{Q}_2^* = (\bar{q}_{2,ij}^*)_{n\times n}$, the comprehensively consistent IVIMLPR is

$$\tilde{R}^* = \begin{pmatrix} ([s_1, s_1], [s_1, s_1]) & ([s_{0.58}, s_{0.89}], [s_{1.05}, s_{1.16}]) & ([s_{0.6}, s_{1.16}], [s_{0.67}, s_{1.34}]) & ([s_{0.35}, s_1], [s_{0.55}, s_{2.08}]) \\ ([s_{1.05}, s_{1.16}], [s_{0.58}, s_{0.89}]) & ([s_1, s_1], [s_1, s_1]) & ([s_{0.87}, s_{1.31}], [s_{0.64}, s_{0.96}]) & ([s_{0.6}, s_{1.12}], [s_{0.52}, s_{1.79}]) \\ ([s_{0.67}, s_{1.34}], [s_{0.6}, s_{1.16}]) & ([s_{0.64}, s_{0.96}], [s_{0.87}, s_{1.31}]) & ([s_1, s_1], [s_1, s_1]) & ([s_{0.58}, s_{0.86}], [s_{0.81}, s_{1.55}]) \\ ([s_{0.55}, s_{2.08}], [s_{0.35}, s_1]) & ([s_{0.52}, s_{1.79}], [s_{0.6}, s_{1.12}]) & ([s_{0.81}, s_{1.55}], [s_{0.58}, s_{0.86}]) & ([s_1, s_1], [s_1, s_1]) \end{pmatrix}$$

Furthermore, the interval-valued intuitionistic multiplicative linguistic priority weight vector is

$$\tilde{w}_1 = ([s_{0.59}, s_{1.01}], [s_{0.79}, s_{1.34}]),\ \tilde{w}_2 = ([s_{0.86}, s_{1.14}], [s_{0.66}, s_{1.11}]),$$
$$\tilde{w}_3 = ([s_{0.71}, s_{1.03}], [s_{0.81}, s_{1.24}]),\ \tilde{w}_2 = ([s_{0.69}, s_{1.55}], [s_{0.59}, s_{0.99}])$$

Step 9: The scores based on formula (6) are $V(\tilde{w}_1) = 0.56$, $V(\tilde{w}_2) = 1.33$, $V(\tilde{w}_3) = 0.72$, $V(\tilde{w}_4) = 1.84$. Thus, the ranking is $x_4 \succ x_2 \succ x_3 \succ x_1$, namely, the fourth coal company is the most suitable partner.

According to the comprehensively consistent IVIMLPR, one can easily check that the ranking of these four coal companies is consistent with IVIMLVs in the comprehensively consistent IVIMLPR $\tilde{R}^*$ obtained from Step 8. It shows that formula (6) is good to calculate the ranking values of objects from consistent IVIMLPR. Furthermore, we can derive the following conclusion to show the rationality of ranking results:

(1) The consistency definition for IVIMLPRs satisfies all properties of the consistency concept for multiplicative linguistic preference relations, which ensures the rationality of judging the consistency of IVIMLPRs;
(2) Missing linguistic judgements are determined based on the consistency analysis, which makes the obtained linguistic judgements have the highest consistency level with the known judgments. This ensures the smallest adjustment for deriving the completely consistent IVIMLPRs. Therefore, it remains the original known information as much as possible;
(3) Models for deriving completely consistent IVIMLPRs own the following three desirable aspects: (i) they are based on associated QMILPRs with the highest

consistency level, which ensures the smallest total adjustment; (ii) they permit the endpoints of IVIMLVs to have different adjustments; (iii) they ensure to derive completely consistent IVIMLPRs from associated consistent QMILPRs;

(4) The weights of the DMs are determined in an objective way that is based on the consensus analysis. The closer the judgments of one DM to those of other DMs are, the larger the weight of him/her will be.

(5) Model (M-7) can achieve the goals listed in the second paragraph on page 20.

All in all, the results obtained from the new method avoid the contradictory situation and own the given consensus level, which ensures the reasonability and reliability.

Notably, IVIMLPRs are a new type of preference relations, and there are no previous methods that can be applied in this example. Thus, we cannot make numerical comparison analysis with previous methods. However, IVIMLPRs as a more general type of preference relations, which can be seen as an extension of several types of linguistic preference relations, such as intuitionistic multiplicative linguistic preference relations, multiplicative interval linguistic preference relations, and multiplicative linguistic preference relations. Therefore, the new method can be directly applied such types of preference relations.

8 Conclusion

To denote the asymmetrically uncertain preferred and non-preferred qualitative judgments of DMs, this paper extended Xu's multiplicative interval linguistic variables (Xu 2006) to introduce IVIMLVs. Then, IVIMLPRs, whose elements are IVIMLVs, are proposed. To derive the rational ranking of objects from IVIMLPRs, this paper has studied the consistency of IVIMLPRs. Based on the defined consistency concept, a mathematical optimization model for judging the consistency of IVIMLPRs is constructed. Meanwhile, mathematical optimization models for determining unknown linguistic variables and deriving consistent IVIMLPRs have been established, respectively. For GDM with IVIMLPRs, we have used individual QMILPRs to define a consensus index and then build a mathematical optimization model to determine the weights of DMs. Furthermore, when the individual consensus level is smaller than the given consensus threshold, a mathematical optimization model for improving the consensus level has been built.

Based on the above developed results, a method for GDM with incomplete and inconsistent IVIMLPRs has been provided, and its application has been shown using a practical GDM problem on evaluating supply chain partners. This paper mainly focused on the theoretical aspect of decision making with IVIMLPRs. In future, we will continue this research by studying other decision-making methods with interval-valued intuitionistic multiplicative linguistic fuzzy information, such as PROMETHEE method, ELECTRE method, TOPSIS method, and VIKOR method. On the other hand, we will further study the application of the new algorithm in other fields, such as evaluating enterprise environment management, medical recommendation, large project management, and risk assessment of complex ecological environment. Although the new method owns several merits, it is based on the assumption that the

DMs completely agree to the consistency and consensus adjustments. However, in some situations, the DMs may conditionally agree with them. Therefore, it is better to include the DMs' opinions in these procedures. Furthermore, the optimization-based procedure seems to a little complex, and we shall continue to study GDM with IVIML-PRs and introduce simpler decision making methods. Moreover, it does not study the associated thresholds.

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