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Partial order relation for approximation operators in covering based rough sets

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ABSTRACT

Covering based rough sets are a generalization of classical rough sets, in which the traditional partition of the universe induced by an equivalence relation is replaced by a covering. Many definitions have been proposed for the lower and upper approximations within this setting. In this paper, we recall the most important ones and organize them into sixteen dual pairs. Then, to provide more insight into their structure, we investigate order relationships that hold among the approximation operators. In particular, we study a point-wise partial order for lower (resp., upper) approximation operators, that can be used to compare their respective approximation fineness. We establish the Hasse diagram for the partial order, showing the relationship between any pair of lower (resp., upper) operators, and identifying its minimal and maximal elements.

1. Introduction

Rough sets, introduced by Pawlak [9] in 1982, provide approximations of concepts in the presence of incomplete information. Essentially, rough set analysis makes statements about the membership of some element x of a non-empty universe set U to the concept of which $A \subseteq U$ is a set of examples, based on the indiscernibility between x and the elements of A. In particular, x belongs to the lower approximation of A if all elements indiscernible from x belong to A, and to the upper approximation of A if at least one element indiscernible from x is a member of A. In Pawlak's original proposal, indiscernibility is modeled by an equivalence relation on U.

Since then, many generalizations of rough set theory have been proposed. A first one is to replace the equivalence relation by a general binary relation. In this case, the binary relation determines collections of sets that no longer form a partition of U [6–8,12,41]. This generalization has been used in applications with incomplete information systems and tables with continuous attributes [4,5,24,26,42]. A second generalization is to replace the partition obtained by the equivalence relation with a covering; i.e., a collection of non-empty sets with union equal to U [14,21,25,36,38,40]. In this paper, we focus on the latter generalization, called covering based rough sets. Some connections between the two generalizations have also been established, for example in [23,29,37,41].

Unlike in classical rough set theory, there is no unique way to define lower and upper approximation operators in covering based rough set theory. In fact, different equivalent characterizations of the classical approximations cease to be

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equivalent when the partition is generalized by a covering. Based on this observation, in [25], Yao and Yao considered twenty pairs of covering based lower and upper approximation operators, where each pair is governed by a duality constraint. Other operators outside this framework appear for example in [21], where Yang and Li present a summary of seven non-dual pairs of approximation operators that were used by Żakowski [27], Pomykala [11], Tsang [15], Zhu [37], Zhu and Wang [39] and Xu and Zhang [20].

In [13], we investigated properties of these operators and some relationships that hold between them, and in particular we proved a characterization of pairs of approximation operators that are at the same time dual and adjoint (i.e., for which there exists a Galois connection). In this paper, we continue this study of covering based approximation operators, focusing on a particular point-wise partial order relation that can be considered among them. Specifically, given two lower approximation operators $\underline{apr}_1$ and $\underline{apr}_2$, and two upper approximation operators $\overline{apr}_1$ and $\overline{apr}_2$, we say that the pair $(\underline{apr}_1, \overline{apr}_1)$ is *finer than* the pair $(\underline{apr}_2, \overline{apr}_2)$ if $\underline{apr}_2(A) \subseteq \underline{apr}_1(A)$ and $\overline{apr}_1(A) \subseteq \overline{apr}_2(A)$ for every set A in U. Clearly, the "is finer than" relation forms a partial order. Moreover, it is very useful in practice, since it helps practitioners in their choice of suitable approximation operators: indeed, a pair of finer operators will allow to approximate a concept more closely from below and from above.

In this paper, we want to establish this partial order for the most commonly used covering based approximation operators, providing an exhaustive evaluation of their pairwise comparability. The remainder of the paper is structured as follows: in Section 2, we present preliminary concepts about classical rough sets, and review lower and upper approximation operators for covering based rough sets that have been proposed in literature. They belong to two main frameworks: the dual framework of Yao and Yao [25], including element based, granule based and system based definitions; and the non-dual framework of Yang and Li [21]. In Section 3, on one hand, we reduce the number of operators by proving some equivalences between them, and on the other hand, we consider some new ones which emerge as duals of the approximation operators considered by Yang and Li. This gives us sixteen pairs of dual and distinct approximation operators. We also list their most important theoretical properties. Section 4 evaluates the fineness order, first for each group of operators separately, and then for all the operators jointly. The central result of our analysis is a Hasse diagram positioning the 16 lower (resp., upper) approximation operators according to the fineness order. We also show the orders for subsets of operators satisfying particular properties, like adjointness and being a meet/join-morphism. Section 5 presents some conclusions and outlines future work.

2. Preliminaries

Throughout this paper, we will assume that U is a finite and non-empty set; $\mathscr{P}(U)$ represents the collection of subsets of U.

2.1. Rough sets

In Pawlak's proposal [9] of rough sets, an approximation space is an ordered pair (U, R), where R is an equivalence relation on U. Yao and Yao [25] consider three different, equivalent ways to define lower and upper approximation operators which are recalled below.

If (U, R) is an approximation space, for each $A \subseteq U$, the *element based*lower and upper approximations of A by R are defined by:

$$\underline{apr}(A) = \{x \in U : [x]_R \subseteq A\} \quad (1)$$
$$\overline{apr}(A) = \{x \in U : [x]_R \cap A \neq \emptyset\} \quad (2)$$

where $[x]_R$ is the equivalence class of x. On the other hand, the *granule based* lower and upper approximations are defined by:

$$\underline{apr}(A) = \bigcup\{[x]_R \in U/R : [x]_R \subseteq A\} \quad (3)$$
$$\overline{apr}(A) = \bigcup\{[x]_R \in U/R : [x]_R \cap A \neq \emptyset\} \quad (4)$$

Finally, the *system based* approximations are obtained from the σ-algebra $\sigma(U/R)$, generated from the equivalence classes, by adding the empty set and making it closed under set union:

$$\underline{apr}(A) = \bigcup\{X \in \sigma(U/R) : X \subseteq A\} \quad (5)$$
$$\overline{apr}(A) = \bigcap\{X \in \sigma(U/R) : X \supseteq A\} \quad (6)$$

When R is not an equivalence relation (U/R is not a partition), these definitions are no longer equivalent. This has inspired the various proposals for covering based approximation operators in the following subsection.

2.2. Covering based rough sets

Covering based rough sets were proposed to extend the range of applications of rough set theory. The basic idea is to replace the partition corresponding to an approximation space by a covering.

Definition 1 [31]. Let $\mathbb{C} = \{K_i\}$ be a family of nonempty subsets of U. $\mathbb{C}$ is called a covering of U if $\cup K_i = U$. The ordered pair $(U, \mathbb{C})$ is called a covering approximation space.

It is clear that a partition generated by an equivalence relation on U is a special case of a covering of U, so the concept of covering is an extension of a partition.

In 1983, Żakowski [27] was the first to propose a pair of lower and upper approximation operators in this setting. Later, Pomykala [11], Yao [22], Couso and Dubois [2], Wybraniec-Skardowska [18], Bonikowski and Bryniarski [1], and Zhu [31–33], among others, defined other approximation operators. In this paper, we consider the following general definition of a lower and an upper approximation operator.

Definition 2. Let $(U, \mathbb{C})$ be a covering approximation space. A function $\underline{apr} : \mathscr{P}(U) \to \mathscr{P}(U)$ is a lower approximation if $\underline{apr}(A) \subseteq A$, for all $A \in \mathscr{P}(U)$. A function $\overline{apr} : \mathscr{P}(U) \to \mathscr{P}(U)$ is an upper approximation if $A \subseteq \overline{apr}(A)$, for all $A \in \mathscr{P}(U)$. The set of all pairs $(\underline{apr}, \overline{apr})$ of a lower and an upper approximation operator is denoted $\mathscr{APR}$.

Although the Eqs. (1)–(6) of lower and upper approximations are equivalent in Pawlak rough set theory, they are no longer equivalent in covering based rough sets. The equivalence classes may be replaced by other types of sets, for example: neighborhoods, minimal and maximal sets of the covering, etc. In the first three subsections below, we review element based, granule based and system based definitions of *dual pairs* of covering based approximation operators, as considered by Yao and Yao [25].

Definition 3. Given a lower (resp., upper) approximation operator apr, its dual is an upper (resp., lower) approximation operator apr^{∂}, defined by $apr^{\partial}(A) = \sim apr(\sim A)$, for all $A \in \mathscr{P}(U)$. A pair $(\underline{apr}, \overline{apr})$ of a lower and an upper approximation is called a dual pair if $\underline{apr} = \overline{apr}^{\partial}$.

A second important branch of covering based approximation operators does not take into account duality; these definitions are recalled in Section 2.2.4.

2.2.1. Element based definition

In rough set theory, the equivalence class of an element $x \in U$ can be considered as its neighborhood, but in covering based rough sets, an element can belong to many sets of the covering $\mathbb{C}$, so we need to define the neighborhood of an element $x \in U$. To this aim, we consider the sets K in $\mathbb{C}$ such that $x \in K$.

Definition 4 [25]. If $\mathbb{C}$ is a covering of U, a neighborhood system for $x \in U, \mathscr{C}(\mathbb{C}, x)$ is defined by:

$$\mathscr{C}(\mathbb{C}, x) = \{K \in \mathbb{C} : x \in K\} \tag{7}$$

Clearly, $\mathscr{C}(\mathbb{C}, x)$ is a subset of $\mathbb{C}$.

Definition 5 [25]. A mapping $N : U \to \mathscr{P}(U)$ is called a neighborhood operator. If $x \in N(x)$ for all $x \in U, N$ is called a reflexive neighborhood operator.

Substituting the equivalence class $[x]_R$ by the neighborhood $N(x)$ in Eqs. (1) and (2), each neighborhood operator defines a pair of approximation operators:

$$\underline{apr}_N(A) = \{x \in U : N(x) \subseteq A\} \tag{8}$$

$$\overline{apr}_N(A) = \{x \in U : N(x) \cap A \neq \emptyset\} \tag{9}$$

We are interested in reflexive neighborhood operators, in this case the operators $\underline{apr}_N$ and $\overline{apr}_N$ satisfy (Theorem 1 in [25]):

$$\underline{apr}_N(A) \subseteq A \subseteq \overline{apr}_N(A). \tag{10}$$

So, $\underline{apr}_N$ can be seen as a lower approximation operator and $\overline{apr}_N$ as an upper approximation operator. Eqs. (8) and (9) give the element based definition in covering based rough set theory, analogously to Eqs. (1) and (2) in rough set theory. From the definitions, it is easy to show that $(\underline{apr}_N, \overline{apr}_N)$ is a dual pair.

Next, we recall some common ways to define neighborhood operators. In a neighborhood system $\mathscr{C}(\mathbb{C}, x)$ the minimal and maximal sets that contain an element $x \in U$ are particularly important.

Definition 6 [1]. Let $(U, \mathbb{C})$ be a covering approximation space and $x \in U$. The set

$$md(\mathbb{C}, x) = \{K \in \mathscr{C}(\mathbb{C}, x) : (\forall S \in \mathscr{C}(\mathbb{C}, x))(S \subseteq K \Rightarrow K = S)\} \tag{11}$$

is called the minimal description of the object x.

By analogy, the notion of maximal description was introduced by Zhu and Wang in [40].

Definition 7 [40]. Let $(U, \mathbb{C})$ be a covering approximation space, $K \in \mathbb{C}$. The set

$$MD(\mathbb{C}, x) = \{K \in \mathscr{C}(\mathbb{C}, x) : (\forall S \in \mathscr{C}(\mathbb{C}, x))(S \supseteq K \Rightarrow K = S)\} \tag{12}$$

is called the maximal description of the object x.

The sets $md(\mathbb{C},x)$ and $MD(\mathbb{C},x)$ represent extreme points of the neighborhood system $\mathscr{C}(\mathbb{C},x)$. For any $K \in \mathscr{C}(\mathbb{C},x)$, we can find sets $K_1 \in md(\mathbb{C},x)$ and $K_2 \in MD(\mathbb{C},x)$ such that $K_1 \subseteq K \subseteq K_2$. From $md(\mathbb{C},x)$ and $MD(\mathbb{C},x)$, it is possible to define the following neighborhood operators [25]:

1. $N_1(x) = \cap\{K : K \in md(\mathbb{C},x)\}$.
2. $N_2(x) = \cup\{K : K \in md(\mathbb{C},x)\}$.
3. $N_3(x) = \cap\{K : K \in MD(\mathbb{C},x)\}$.
4. $N_4(x) = \cup\{K : K \in MD(\mathbb{C},x)\}$.

The set $N_1(x)$ equals $\cap md(\mathbb{C},x)$, for each $x \in U$, and is called the minimal neighborhood of x. It satisfies some important properties as is shown in the following proposition.

Proposition 1 [21]. *Let $\mathbb{C}$ be a covering of U and $K \in \mathbb{C}$, then*

a. $K = \cup_{x\in K} N_1(x)$.
b. *If $y \in N_1(x)$ then $N_1(y) \subseteq N_1(x)$, for x, y in U.*

2.2.2. Granule based definition

The lower and upper approximation operators from a covering $\mathbb{C}$ are defined, according to the granule based definition [25], by:

$$\underline{apr}'_{\mathbb{C}}(A) = \bigcup\{K \in \mathbb{C} : K \subseteq A\} = \{x \in U : (\exists K \in \mathbb{C})(x \in K \wedge K \subseteq A)\} \tag{13}$$

$$\overline{apr}''_{\mathbb{C}}(A) = \bigcup\{K \in \mathbb{C} : K \cap A \neq \emptyset\} = \{x \in U : (\exists K \in \mathbb{C})(x \in K \wedge K \cap A \neq \emptyset)\} \tag{14}$$

It can be checked that $\overline{apr}'_{\mathbb{C}}$ and $\overline{apr}''_{\mathbb{C}}$ are not dual; therefore, in [25], the dual approximation operators corresponding to Eqs. (13) and (14) were considered. They are defined by:

$$\overline{apr}'_{\mathbb{C}}(A) = \sim \underline{apr}'_{\mathbb{C}}(\sim A) = \{x \in U : (\forall K \in \mathbb{C})(x \in K \Rightarrow K \cap A \neq \emptyset)\} \tag{15}$$

$$\underline{apr}''_{\mathbb{C}}(A) = \sim \overline{apr}''_{\mathbb{C}}(\sim A) = \{x \in U : (\forall K \in \mathbb{C})(x \in K \Rightarrow K \subseteq A)\} \tag{16}$$

Apart from using $\mathbb{C}$ directly in the above equations, it is also possible to derive new coverings of U from $\mathbb{C}$. In [25], the following coverings are considered:

1. $\mathbb{C}_1 = \cup\{md(\mathbb{C},x) : x \in U\}$.
2. $\mathbb{C}_2 = \cup\{MD(\mathbb{C},x) : x \in U\}$.
3. $\mathbb{C}_3 = \{\cap(md(\mathbb{C},x)) : x \in U\} = \{\cap(\mathscr{C}(\mathbb{C},x)) : x \in U\}$.
4. $\mathbb{C}_4 = \{\cup(MD(\mathbb{C},x)) : x \in U\} = \{\cup(\mathscr{C}(\mathbb{C},x)) : x \in U\}$.

For example, the covering $\mathbb{C}_1$ is the collection of all sets in the minimal description of each $x \in U$, while $\mathbb{C}_3$ is the collection of the intersections of minimal descriptions for each $x \in U$, i.e., $\{N_1(x) : x \in U\}$. Each covering defines two dual pairs of approximation operators given by Eqs. (13), (15), (14) and (16), respectively.

Another way to derive a new covering from $\mathbb{C}$ is based on a reduction process. Since a partition of U consists of pairwise disjoint sets, in a covering we can consider to eliminate sets which are intersections or unions of other sets in $\mathbb{C}$.

Definition 8 [25]. Let $\mathbb{F}$ be a family of non-empty subsets of U. If $K \in \mathbb{F}$ is the intersection of some sets in $\mathbb{F} \setminus K$, then K is said to be intersection reducible in $\mathbb{F}$, otherwise K is called intersection irreducible. If K is the union of some sets in $\mathbb{F} \setminus K$, then K is said to be union reducible in $\mathbb{F}$, otherwise K is called union irreducible.

Definition 9 [25]. Let $(U,\mathbb{C})$ be a covering approximation space. The set $\mathbb{C}_{\cap}$ of all intersection irreducible elements of $\mathbb{C}$ is a covering that is called the intersection reduct of $\mathbb{C}$. The set $\mathbb{C}_{\cup}$ of all union irreducible elements of $\mathbb{C}$ is a covering that is called the union reduct of $\mathbb{C}$.

Again, the coverings $\mathbb{C}_{\cap}$ and $\mathbb{C}_{\cup}$ define new lower and upper approximation operators using Eqs. (13) and (14) and their duals in Eqs. (15) and (16).

2.2.3. System based definition

In order to generalize the system based definition in Eqs. (5) and (6), Yao and Yao considered the notion of a closure system.

Definition 10 [25]. A family of subsets of U is called a closure system over U if it contains U and is closed under set intersection.

Given a closure system $\mathbb{S}$, one can construct its dual system $\mathbb{S}'$, with the complements of each X in $\mathbb{S}$, as follows:

$$\mathbb{S}' = \{\sim X : X \in \mathbb{S}\} \tag{17}$$

The system $\mathbb{S}'$ contains $\emptyset$ and it is closed under set union.

Definition 11 [25]. Let $\mathbb{C}$ be a covering of U. The intersection closure of $\mathbb{C}$, denoted by $\cap$-closure($\mathbb{C}$), is the minimum subset of $\mathscr{P}(U)$ that contains $\mathbb{C}, \emptyset$ and U, and is closed under set intersection. The union closure of $\mathbb{C}$, denoted by $\cup$-closure($\mathbb{C}$), is the minimum subset of $\mathscr{P}(U)$ that contains $\mathbb{C}, \emptyset$ and U, and is closed under set union.

Approximation operators with respect to the system based definition are defined by using a pair of a closure system and its respective dual system.

$$\begin{aligned} S_\cap &= \{(\cap\text{-closure}(\mathbb{C}))', \cap\text{-closure}(\mathbb{C})\} \\ S_\cup &= \{\cup\text{-closure}(\mathbb{C}), (\cup\text{-closure}(\mathbb{C}))'\} \end{aligned} \tag{18}$$

The lower approximation operators for $A \subseteq U$ are defined by:

$$\underline{apr}_{S_\cap}(A) = \bigcup\{X \in (\cap\text{-closure}(\mathbb{C}))' : X \subseteq A\} \tag{19}$$

$$\underline{apr}_{S_\cup}(A) = \bigcup\{X \in \cup\text{-closure}(\mathbb{C}) : X \subseteq A\} \tag{20}$$

Dually, the upper approximation operators for $A \subseteq U$ are defined by:

$$\overline{apr}_{S_\cap}(A) = \bigcap\{X \in \cap\text{-closure}(\mathbb{C}) : X \supseteq A\} \tag{21}$$

$$\overline{apr}_{S_\cup}(A) = \bigcap\{X \in (\cup\text{-closure}(\mathbb{C}))' : X \supseteq A\} \tag{22}$$

Eqs. (19) and (21) define a dual pair of approximation operators: $(\underline{apr}_{S_\cap}, \overline{apr}_{S_\cap})$. On the other hand, Eqs. (20) and (22) define another dual pair of approximation operators: $(\underline{apr}_{S_\cup}, \overline{apr}_{S_\cup})$.

Remark 1. Among all the dual pairs considered by Yao and Yao in [25], $(\underline{apr}_{S_\cap}, \overline{apr}_{S_\cap})$ stands out because it is the only one that, when the covering $\mathbb{C}$ is a partition, does not obtain the same results as Pawlak's approximation operators.

2.2.4. Non-dual framework of lower and upper approximations

A summary of seven pairs of non dual approximation operators within a covering approximation space $(U, \mathbb{C})$, was presented in [16,21]. These pairs consist of seven different upper approximation operators, combined with two lower approximation operators, which are recalled first. Let $A \in \mathscr{P}(U)$.

- $L_1^{\mathbb{C}}(A) = \cup\{K \in \mathbb{C} : K \subseteq A\}$.
- $L_2^{\mathbb{C}}(A) = \cup\{N_1(x) : N_1(x) \subseteq A\}$.

$L_1^{\mathbb{C}}$ is the same definition used by Yao and Yao [25] for $\underline{apr}'_{\mathbb{C}}$ and $L_2^{\mathbb{C}}$ is the particular case of $L_1^{\mathbb{C}}$, when we use $\mathbb{C}_3$ instead of $\mathbb{C}$, so $L_2^{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_3}$.

The seven upper approximation operators are defined as follows, for $A \in \mathscr{P}(U)$:

- $H_1^{\mathbb{C}}(A) = L_1^{\mathbb{C}}(A) \cup (\cup\{md(\mathbb{C}, x) : x \in A \setminus L_1^{\mathbb{C}}(A)\})$.
- $H_2^{\mathbb{C}}(A) = \cup\{K \in \mathbb{C} : K \cap A \neq \emptyset\} = \overline{apr}''_{\mathbb{C}}(A)$.
- $H_3^{\mathbb{C}}(A) = \cup\{md(\mathbb{C}, x) : x \in A\}$.
- $H_4^{\mathbb{C}}(A) = L_1^{\mathbb{C}}(A) \cup (\cup\{K : K \cap (A \setminus L_1^{\mathbb{C}}(A)) \neq \emptyset\})$.
- $H_5^{\mathbb{C}}(A) = \cup\{N_1(x) : x \in A\}$.
- $H_6^{\mathbb{C}}(A) = \{x : N_1(x) \cap A \neq \emptyset\} = \overline{apr}_{N_1}(A)$.
- $H_7^{\mathbb{C}}(A) = \cup\{N_1(x) : N_1(x) \cap A \neq \emptyset\} = \overline{apr}''_{\mathbb{C}_3}(A)$.

$H_1^{\mathbb{C}}$ was originally proposed by Żakowski [27], while $H_2^{\mathbb{C}}$ is due to Pomykala [11]. Tsang et al. [15] studied $H_3^{\mathbb{C}}$, Zhu et al. [17,39,37] defined $H_4^{\mathbb{C}}$ and $H_5^{\mathbb{C}}$, Xu and Wang [19] introduced $H_6^{\mathbb{C}}$, and finally Xu and Zhang $H_7^{\mathbb{C}}$ [20].

The first four upper approximations were studied in conjunction with $L_1^{\mathbb{C}}$, while the last three were paired with $L_2^{\mathbb{C}}$. It can be checked that none of the thus generated pairs is dual.

Also, we adopted the notations given by Yang and Li in [21], because there is no uniform notation for them in literature. For example, in Zhang et al. [28], *TH* refers to the second upper approximation operator, while in [33] *FH*, refers to the same operator. Finally, note that some of the above operators also appear in Yao and Yao's framework.

3. Dual pairs of approximation operators and their properties

In this section, based on the operators discussed in Section 2.2, we compile a list of dual pairs of approximation operators to be considered in our study on order relations. On one hand, this list includes the dual pairs proposed by Yao and Yao in [25] and discussed in Sections 2.2.1, 2.2.2 and 2.2.3. As we will see below, some of them are equivalent, hence the total number of pairs can be reduced. On the other hand, we add to the list those pairs obtained by coupling $H_1^{\mathbb{C}}, H_3^{\mathbb{C}}, H_4^{\mathbb{C}}$ and $H_5^{\mathbb{C}}$ from Section 2.2.4 with their respective dual lower approximations.

Some equivalences were already established in [13], and are summarized in the following proposition.

Proposition 2 [13].

a. $\underline{apr}'_{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_1} = \underline{apr}'_{\mathbb{C}_\cup} = \underline{apr}'_{S_\cup}$
$\overline{apr}'_{\mathbb{C}} = \overline{apr}'_{\mathbb{C}_1} = \overline{apr}'_{\mathbb{C}_\cup} = \overline{apr}'_{S_\cup}$.
b. $\underline{apr}''_{\mathbb{C}} = \underline{apr}''_{\mathbb{C}_2} = \underline{apr}''_{\mathbb{C}_\cap}$
$\overline{apr}''_{\mathbb{C}} = \overline{apr}''_{\mathbb{C}_2} = \overline{apr}''_{\mathbb{C}_\cap}$.
c. $\underline{apr}'_{\mathbb{C}_3} = \underline{apr}_{N_1}$
$\overline{apr}'_{\mathbb{C}_3} = \overline{apr}_{N_1}$.

This already reduces the twenty dual pairs considered by Yao and Yao to fourteen. The following propositions show two further equivalences.

Proposition 3. $\mathbb{C}_1 = \mathbb{C}_\cup$.

Proof. We will show that $\mathbb{C}_1 \subseteq \mathbb{C}_\cup$ and $\mathbb{C}_\cup \subseteq \mathbb{C}_1$.

Let us suppose that $K \in \mathbb{C}_1$ and that K is union reducible, that is, $K \in md(\mathbb{C}, x)$ for some $x \in U$ and $K = K_1 \cup K_2 \cup \ldots \cup K_l$ with $K_i \in \mathbb{C}$ and $K_i \neq K$, for $i = 1, \ldots, l$. We have $x \in K$, therefore there exists a $j \in \{1, 2, \ldots, l\}$ such that $x \in K_j \subset K$. Hence, $K \notin md(\mathbb{C}, x)$. This is a contradiction, so K must be union irreducible, so $K \in \mathbb{C}_\cup$.

On the other hand, if $K \notin \mathbb{C}_1$, then for all $x \in U, K \notin md(\mathbb{C}, x)$. In particular, let $x \in K$. Since $K \notin md(\mathbb{C}, x)$, there exists $K_0^x \in md(\mathbb{C}, x)$ such that $x \in K_0^x \subset K$. So, we have $K = \cup_{x \in K}\{x\} \subseteq \cup_{x \in K} K_0^x \subseteq K$, so $K = \cup K_0^x$, hence K is reducible and $K \notin \mathbb{C}_\cup$. □

Corollary 1. $\underline{apr}''_{\mathbb{C}_1} = \underline{apr}''_{\mathbb{C}_\cup}$.

Proposition 4. $\underline{apr}''_{\mathbb{C}_2} = \underline{apr}_{N_4}$.

Proof. Let $x \in U$ and $A \in \mathscr{P}(U)$. It holds that $x \in \underline{apr}_{N_4}(A) \iff (\forall K \in MD(\mathbb{C}, x))(K \subseteq A)$ and $x \in \underline{apr}''_{\mathbb{C}_2}(A) \iff (\forall K \in \cup\{MD(\mathbb{C}, y) : y \in U\})(x \in K \Rightarrow K \subseteq A)$.

Clearly, if $x \in \underline{apr}''_{\mathbb{C}_2}(A)$, then $x \in \underline{apr}_{N_4}(A)$, so $\underline{apr}''_{\mathbb{C}_2}(A) \subseteq \underline{apr}_{N_4}(A)$.

On the other hand, suppose $x \in \underline{apr}_{N_4}(A)$ and $x \notin \underline{apr}''_{\mathbb{C}_2}(A)$. Let $y \in U$ and $K \in MD(\mathbb{C}, y)$ such that $x \in K$ and $K \nsubseteq A$. Then $K \notin MD(\mathbb{C}, x)$, so there exists $S \in MD(\mathbb{C}, x)$ such that $K \subset S$ and $S \subseteq A$. But then $K \subseteq A$ as well, which is a contradiction. In other words, $\underline{apr}_{N_4}(A) \subseteq \underline{apr}''_{\mathbb{C}_2}(A)$. □

It can be checked that no further identities hold among the approximation operators considered by Yao and Yao [25] and those considered by Yang and Li [21]. Hence, there are sixteen groups of different dual pairs of approximations operators, which are listed in Table 1.

As mentioned in the introduction, the main objective of this paper will be to establish a point-wise partial order for the lower and upper approximation operators in this table, comparing them according to the fineness of their approximations. At the same time, we can also differentiate between the approximation operators according to the theoretical properties they satisfy. Table 2 lists five important properties, all of which hold in an approximation space in Pawlak's sense, and points out which of the groups in Table 1 satisfy them. The proofs of most of these properties can be reconstructed from literature, see e.g. [13,31–35,37], taking into account that fourteen out of the sixteen dual pairs of operators can be expressed by means of $L_1^{\mathbb{C}}, L_2^{\mathbb{C}}, H_i^{\mathbb{C}}$ and their respective dual operators. The remaining proofs can be established by simple verification, and counterexamples are easy to find for the negative results. It is interesting to note that none of the currently considered groups satisfies all properties; in particular, the properties of adjointness and idempotence are never simultaneously satisfied.

4. Partial order relation for approximation operators

In this section, we systematically investigate a point-wise partial order relation among pairs of lower and upper approximation operators. This partial order is defined as follows:

Table 1
List of different dual pairs of lower and upper approximations.

Number	Lower approximation	Upper approximation
1	$\underline{apr}_{N_1} = \underline{apr}'_{\mathbb{C}_3} = (H_6^{\mathbb{C}})^{\partial} = L_2^{\mathbb{C}}$	$\overline{apr}_{N_1} = \overline{apr}'_{\mathbb{C}_3} = H_6^{\mathbb{C}}$
2	$\underline{apr}_{N_2}$	$\overline{apr}_{N_2}$
3	$\underline{apr}_{N_3}$	$\overline{apr}_{N_3}$
4	$\underline{apr}_{N_4} = \underline{apr}''_{\mathbb{C}} = \underline{apr}''_{\mathbb{C}_2} = \underline{apr}''_{\mathbb{C}_\cap} = \left(H_2^{\mathbb{C}}\right)^{\partial}$	$\overline{apr}_{N_4} = \overline{apr}''_{\mathbb{C}} = \overline{apr}''_{\mathbb{C}_2} = \overline{apr}''_{\mathbb{C}_\cap} = H_2^{\mathbb{C}}$
5	$\underline{apr}'_{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_1} = \underline{apr}'_{\mathbb{C}_\cup} = \underline{apr}_{S_\cup} = L_1^{\mathbb{C}}$	$\overline{apr}'_{\mathbb{C}} = \overline{apr}'_{\mathbb{C}_1} = \overline{apr}'_{\mathbb{C}_\cup} = \overline{apr}_{S_\cup}$
6	$\underline{apr}'_{\mathbb{C}_2}$	$\overline{apr}'_{\mathbb{C}_2}$
7	$\underline{apr}'_{\mathbb{C}_4}$	$\overline{apr}'_{\mathbb{C}_4}$
8	$\underline{apr}'_{\mathbb{C}_\cap}$	$\overline{apr}'_{\mathbb{C}_\cap}$
9	$\underline{apr}''_{\mathbb{C}_1} = \underline{apr}''_{\mathbb{C}_\cup}$	$\overline{apr}''_{\mathbb{C}_1} = \overline{apr}''_{\mathbb{C}_\cup}$
10	$\underline{apr}''_{\mathbb{C}_3} = \left(H_7^{\mathbb{C}}\right)^{\partial}$	$\overline{apr}''_{\mathbb{C}_3} = H_7^{\mathbb{C}}$
11	$\underline{apr}''_{\mathbb{C}_4}$	$\overline{apr}''_{\mathbb{C}_4}$
12	$\underline{apr}_{S_\cap}$	$\overline{apr}_{S_\cap}$
13	$\left(H_1^{\mathbb{C}}\right)^{\partial}$	$H_1^{\mathbb{C}}$
14	$\left(H_3^{\mathbb{C}}\right)^{\partial}$	$H_3^{\mathbb{C}}$
15	$\left(H_4^{\mathbb{C}}\right)^{\partial}$	$H_4^{\mathbb{C}}$
16	$\left(H_5^{\mathbb{C}}\right)^{\partial}$	$H_5^{\mathbb{C}}$

Definition 12. Let $\underline{apr}_1$ and $\underline{apr}_2$ be two lower approximation operators, and $\overline{apr}_1$ and $\overline{apr}_2$ two upper approximation operators. We write:

- $\underline{apr}_1 \leq_l \underline{apr}_2$, if and only if $\underline{apr}_1(A) \subseteq \underline{apr}_2(A)$, for all $A \subseteq U$.
- $\overline{apr}_1 \leq_u \overline{apr}_2$, if and only if $\overline{apr}_1(A) \subseteq \overline{apr}_2(A)$, for all $A \subseteq U$.
- $(\underline{apr}_1, \overline{apr}_1) \leqslant (\underline{apr}_2, \overline{apr}_2)$, if and only if $\underline{apr}_1 \geqslant_l \underline{apr}_2$ and $\overline{apr}_1 \leq_u \overline{apr}_2$.

It is easy to see that $\leq_l, \leq_u$ and $\leqslant$ are indeed reflexive, anti-symmetric and transitive. Moreover, it is easy to verify that the partial order $\leqslant$, which may be read as "is finer than" forms a bounded lattice on the set $\mathscr{APR}$ of pairs of approximation operators, with smallest element $(\underline{apr}, \overline{apr})$ where $\underline{apr}(A) = \overline{apr}(A) = A$, and largest element $(\underline{apr}, \overline{apr})$ where $\underline{apr}(A) = \emptyset$ and $\overline{apr}(A) = U$, for any $A \subseteq U$. The lattice meet operation is defined as: $(\underline{apr}_1, \overline{apr}_1) \cap (\underline{apr}_2, \overline{apr}_2) = (\underline{apr}, \overline{apr})$ where $\underline{apr}(A) = \underline{apr}_1(A) \cup \underline{apr}_2(A)$ and $\overline{apr}(A) = \overline{apr}_1(A) \cap \overline{apr}_2(A)$, with $A \subseteq U$. The lattice join operation is defined dually.

Remark 2. In a natural way, it is possible to consider a second partial order on $\mathscr{APR}$, by defining $(\underline{apr}_1, \overline{apr}_1) \leqslant' (\underline{apr}_2, \overline{apr}_2)$ if and only if $\underline{apr}_1 \leq_l \underline{apr}_2$ and $\overline{apr}_1 \leq_u \overline{apr}$. The partial order $\leqslant'$ forms a bounded lattice on $\mathscr{APR}$, with smallest element $(\underline{apr}, \overline{apr})$ where $\underline{apr}(A) = \emptyset$ and $\overline{apr}(A) = A$, and largest element $(\underline{apr}, \overline{apr})$ where $\underline{apr}(A) = A$ and $\overline{apr}(A) = U$, for all $A \subseteq U$. The lattice meet operation is defined as: $(\underline{apr}_1, \overline{apr}_1) \cap' (\underline{apr}_2, \overline{apr}_2) = (\underline{apr}, \overline{apr})$ where $\underline{apr}(A) = \underline{apr}_1(A) \cup \underline{apr}_2(A)$ and $\overline{apr}(A) = \overline{apr}_1(A) \cup \overline{apr}_2(A)$, with $A \subseteq U$. Again, the join operation is defined dually. In the context of bilattice theory [3], $\leqslant$is called the knowledge order on $\mathscr{APR}$ and $\leqslant'$ is called its truth order.

For practical purposes, the partial order $\leqslant$is particularly relevant, since it allows us to compare pairs of approximation operators in terms of their suitability for data analysis. In particular, the definitions of accuracy and quality of classification provided for Pawlak's rough sets [10] can be generalized to covering based rough sets.

Table 2
Evaluation of properties of covering based rough sets.

Name	Property	Satisfied by
Adjointness	$\overline{apr}(A) \subseteq B \Longleftrightarrow A \subseteq \underline{apr}(B)$	$4, 9, 10, 11$
Monotonicity	$A \subseteq B \Rightarrow \underline{apr}(A) \subseteq \underline{apr}(B)$ $A \subseteq B \Rightarrow \overline{apr}(A) \subseteq \overline{apr}(B)$	All groups, except 13 and 15
Meet/join-morphism	$\underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B)$ $\overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B)$	$1, 2, 3, 4, 9, 10, 11, 14, 16$
Idempotence	$\underline{apr}(\underline{apr}(A)) = \underline{apr}(A)$	$1, 5, 6, 7, 8, 13, 15, 16$
$\emptyset$ and U	$\underline{apr}(U) = U$ $\overline{apr}(\emptyset) = \emptyset$	All groups

Definition 13. If $(\underline{apr}, \overline{apr})$ is a pair of a lower and an upper approximation operator, the accuracy of $A \subseteq U$ is defined as:

$$\alpha_{\underline{apr}}^{\overline{apr}}(A) = \frac{|\underline{apr}(A)|}{|\overline{apr}(A)|} \tag{23}$$

On the other hand, the quality of classification of $A \subseteq U$, by means of $\underline{apr}$, is defined as:

$$\gamma_{\underline{apr}}(A) = \frac{|\underline{apr}(A)|}{|A|} \tag{24}$$

The quality of classification of a subset $A \subseteq U$ can be extended to a partition $Y = \{Y_1, \ldots, Y_n\}$ of U:

$$\gamma_{\underline{apr}}(Y) = \sum \frac{|\underline{apr}(Y_i)|}{|U|} \tag{25}$$

$\gamma_{\underline{apr}}(Y)$ can be seen as the ratio of elements of U that can be classified with certainty into one of the classes of Y. Clearly, it is desirable to have $\gamma_{\underline{apr}}(Y)$ as high as possible. The following proposition shows the relationship with the partial order $\leqslant$.

Proposition 5. *If $(\underline{apr}_1, \overline{apr}_1)$ and $(\underline{apr}_2, \overline{apr}_2)$ are two pairs of approximation operators such that $(\underline{apr}_1, \overline{apr}_1) \leqslant (\underline{apr}_2, \overline{apr}_2)$, then $\alpha_{\underline{apr}_2}^{\overline{apr}_2}(A) \leqslant \alpha_{\underline{apr}_1}^{\overline{apr}_1}(A)$ and $\gamma_{\underline{apr}_2}(A) \leqslant \gamma_{\underline{apr}_1}(A)$ for all $A \subseteq U$.*

Proof. Let $A \subseteq U$. If $(\underline{apr}_1, \overline{apr}_1) \leqslant (\underline{apr}_2, \overline{apr}_2)$ then $\underline{apr}_2(A) \subseteq \underline{apr}_1(A)$ and $\overline{apr}_1(A) \subseteq \overline{apr}_2(A)$. Therefore $|\underline{apr}_2(A)| \leqslant |\underline{apr}_1(A)|$ and $|\overline{apr}_1(A)| \leqslant |\overline{apr}_2(A)|$. So $\alpha_{\underline{apr}_2}^{\overline{apr}_2}(A) = \frac{|\underline{apr}_2(A)|}{|\overline{apr}_2(A)|} \leqslant \frac{|\underline{apr}_1(A)|}{|\overline{apr}_1(A)|} = \alpha_{\underline{apr}_1}^{\overline{apr}_1}(A)$. The inequality $\gamma_{\underline{apr}_2}(A) \leqslant \gamma_{\underline{apr}_1}(A)$ can be established similarly. □

In the remainder of this section, we will be concerned with dual pairs of approximation operators. In this case, the following result can be established.

Proposition 6. *Let $(\underline{apr}_1, \overline{apr}_1)$ and $(\underline{apr}_2, \overline{apr}_2)$ be two dual pairs of approximation operators. It holds that*

$$(\underline{apr}_1, \overline{apr}_1) \leqslant (\underline{apr}_2, \overline{apr}_2) \iff \underline{apr}_1 \geqslant_l \underline{apr}_2 \iff \overline{apr}_2 \leq_u \overline{apr}_1. \tag{26}$$

Proof. Direct from the definition of duality and the partial orders. □

In other words, in order to establish the partial order for dual pairs of approximation operators, it suffices to know the partial order $\leqslant_l$ for lower approximation operators, as the partial order $\leqslant_u$ for upper approximation operators can be obtained with the reverse partial order of its duals. From now on, to simplify the notation, we will refer to both $\leqslant_l$ and $\leqslant_u$ by $\leqslant$.

In the following subsections, we first evaluate the order relationships that hold between elements of different groups of approximation operators: element based, granule based and system based definitions of Yao and Yao [25], and upper approximation operators of Yang and Li [21]. Afterwards, we combine these results to construct an integrated Hasse diagram for all the operators considered in Table 1.

4.1. Partial order for element based definitions

The following propositions establish the relationship among element based approximation operators, defined in Eqs. (8) and (9) using neighborhood operators.

Proposition 7. *If N and N' are neighborhood operators such that $N(x) \subseteq N'(x)$ for all $x \in U$, then $\underline{apr}_{N'} \leqslant \underline{apr}_N$.*

Proof. We will show that $\underline{apr}_{N'}(A) \subseteq \underline{apr}_N(A)$, for any $A \subseteq U$. If $x \in \underline{apr}_{N'}(A), N'(x) \subseteq A$, hence $N(x) \subseteq N'(x) \subseteq A$ for all $x \in U$, so $x \in \underline{apr}_N(A)$. □

Proposition 8. *For $x \in U$, it holds that $N_1(x) \subseteq N_2(x), N_3(x) \subseteq N_4(x), N_1(x) \subseteq N_3(x)$ and $N_2(x) \subseteq N_4(x)$.*

Proof. The first two inclusions follow directly from the definition of neighborhood systems. For the third one, we can see that for each $K \in N_1(x)$ there exists $K' \in N_3(x)$ such that $K \subseteq K'$. So, $\cap\{K \in md(\mathbb{C}, x)\} \subseteq \cap\{K' \in MD(\mathbb{C}, x)\}$, from which follows $N_1(x) \subseteq N_3(x)$. The final inclusion can be proved similarly. □

Proposition 9.

a. $\underline{apr}_{N_4} \leqslant \underline{apr}_{N_2} \leqslant \underline{apr}_{N_1}$.
b. $\underline{apr}_{N_4} \leqslant \underline{apr}_{N_3} \leqslant \underline{apr}_{N_1}$.

Proof. Direct from Propositions 7 and 8. □

Moreover, $\underline{apr}_{N_2}$ and $\underline{apr}_{N_3}$ are not comparable, as we can see in Example 1 below.

Example 1. For simplicity, we use a special notation for sets and collections, for example the set $\{1,2,3\}$ will be written as 123 and the collection $\{\{1,2,3\},\{3,4\}\}$ will be written as $\{123, 34\}$. Let us consider the covering $\mathbb{C} = \{1, 23, 123, 34\}$ of $U = 1234$. The neighborhood system $\mathscr{C}(\mathbb{C}, x)$, the minimal description $md(\mathbb{C}, x)$, the maximal description $MD(\mathbb{C}, x)$ and the four neighborhood operators obtained from $\mathscr{C}(\mathbb{C}, x)$ are listed in Table 3.

From the neighborhoods in Table 3, the lower approximations of $A = 23$ are: $\underline{apr}_{N_1}(A) = 23$, $\underline{apr}_{N_2}(A) = 2$, $\underline{apr}_{N_3}(A) = 3$ and $\underline{apr}_{N_4}(A) = \emptyset$. In this example, we can see that $\underline{apr}_{N_2} \not\leqslant \underline{apr}_{N_3}$ and $\underline{apr}_{N_3} \not\leqslant \underline{apr}_{N_2}$, so these operators are not comparable.

Using Propositions 7 and 8 and Example 1, we can establish the partial order for the lower approximation operators in this section; the partial order for the upper approximations follows from Proposition 6. The corresponding Hasse diagrams are shown in Fig. 1. The order relation $apr_{N_i} \leqslant apr_{N_j}$ is represented by means of an arrow from apr_{N_i} to apr_{N_j}.

4.2. Partial order for granule based definitions

The granule based approximation operators definitions were presented in Eqs. (13)–(16). In this section, we will evaluate the order relation for approximation operators related with the coverings $\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4$ and $\mathbb{C}_\cap$ (recall that by Proposition 2, $\underline{apr}'_{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_1}$ and $\underline{apr}''_{\mathbb{C}} = \underline{apr}''_{\mathbb{C}_\cap}$, and that by Proposition 3, $\mathbb{C}_\cup = \mathbb{C}_1$). First, Propositions 10 and 11 establish a general order relation for granule based lower approximation operators $\underline{apr}'$.

Proposition 10. *If* $\mathbb{C}$ *and* $\mathbb{C}'$ *are coverings of U such that* $\mathbb{C} \subseteq \mathbb{C}'$, *then* $\underline{apr}'_{\mathbb{C}} \leqslant \underline{apr}'_{\mathbb{C}'}$.

Proof. Since $\underline{apr}'_{\mathbb{C}}(A) = \cup\{K \in \mathbb{C} : K \subseteq A\}$ and $\mathbb{C} \subseteq \mathbb{C}'$, we have $\cup\{K \in \mathbb{C} : K \subseteq A\} \subseteq \cup\{K \in \mathbb{C}' : K \subseteq A\}$. Then $\underline{apr}'_{\mathbb{C}}(A) \subseteq \underline{apr}'_{\mathbb{C}'}(A)$, for all $A \subseteq U$ and $\underline{apr}'_{\mathbb{C}} \leqslant \underline{apr}'_{\mathbb{C}'}$. □

Proposition 11. *If* $\mathbb{C}$ *and* $\mathbb{C}'$ *are coverings of U such that, for all* $K \in \mathbb{C}, K = \bigcup_{\alpha \in I} L_\alpha$ *for* $(L_\alpha)_{\alpha \in I} \subseteq \mathbb{C}'$, *then* $\underline{apr}'_{\mathbb{C}} \leqslant \underline{apr}'_{\mathbb{C}'}$.

Proof. If $x \in \underline{apr}'_{\mathbb{C}}(A)$, then there exists a $K_0 \in \mathbb{C}$ such that $x \in K_0 \subseteq A$. But $x \in K_0 = \bigcup_{\alpha \in I} L_\alpha \subseteq A$, with $(L_\alpha)_{\alpha \in I} \subseteq \mathbb{C}'$. Hence $x \in L_\alpha \subseteq A$, for some α in I, therefore $x \in \underline{apr}'_{\mathbb{C}'}(A)$. □

Proposition 12. *Let* $\mathbb{C}$ *be a covering of U. It holds that:*

$$\underline{apr}'_{\mathbb{C}_4} \leqslant \underline{apr}'_{\mathbb{C}_2} \leqslant \underline{apr}'_{\mathbb{C}_\cap} \leqslant \underline{apr}'_{\mathbb{C}_1} \leqslant \underline{apr}'_{\mathbb{C}_3}.$$

Proof. It is easy to verify that the pairs of coverings $\mathbb{C}_4 - \mathbb{C}_2$, $\mathbb{C}_2 - \mathbb{C}_1$ and $\mathbb{C}_1 - \mathbb{C}_3$ satisfy the conditions of Proposition 11. For example, for coverings $\mathbb{C}_4 - \mathbb{C}_2$, we have: $\mathbb{C}_4 = \{\cup MD(\mathbb{C}, x) : x \in U\}$ and $\mathbb{C}_2 = \cup\{MD(\mathbb{C}, x) : x \in U\}$, clearly they satisfy the conditions of Proposition 11. Hence, $\underline{apr}'_{\mathbb{C}_4} \leqslant \underline{apr}'_{\mathbb{C}_2} \leqslant \underline{apr}'_{\mathbb{C}_1} \leqslant \underline{apr}'_{\mathbb{C}_3}$.

To see that $\underline{apr}'_{\mathbb{C}_2} \leqslant \underline{apr}_{\mathbb{C}_\cap}$, we prove that $\mathbb{C}_2 \subseteq \mathbb{C}_\cap$. The result then follows from Proposition 10. If $K \in \mathbb{C}_2, K \in MD(\mathbb{C}, x_0)$ for some $x_0 \in U$. If $K = \bigcap_{i \in I} K_i$ for $(K_i)_{i \in I} \subseteq \mathbb{C} - \{K\}$, then $K \subseteq K_i$ for all i in I, so $K \notin MD(\mathbb{C}, x_0)$ which is a contradiction. Hence, $K \in \mathbb{C}_\cap$.

Finally, we prove that $\underline{apr}'_{\mathbb{C}_\cap} \leqslant \underline{apr}'_{\mathbb{C}_1}$. First of all, it is easy to see that $\mathbb{C}_\cap \subseteq \mathbb{C}$, so by Proposition 10, we have $\underline{apr}'_{\mathbb{C}_\cap} \leqslant \underline{apr}'_{\mathbb{C}}$. From Proposition 2, we know that $\underline{apr}'_{\mathbb{C}_1} = \underline{apr}'_{\mathbb{C}}$, so $\underline{apr}'_{\mathbb{C}_\cap} \leqslant \underline{apr}'_{\mathbb{C}_1}$. □

The following proposition establishes a general order relation for granule based upper approximation operators $\overline{apr}''_{\mathbb{C}}$.

Proposition 13. *If* $\mathbb{C}$ *and* $\mathbb{C}'$ *are coverings of U such that, for all* $K \in \mathbb{C}$, *there exists* $L \in \mathbb{C}'$ *such that* $K \subseteq L$, *then* $\overline{apr}''_{\mathbb{C}} \leqslant \overline{apr}''_{\mathbb{C}'}$.

Proof. If $x \in \overline{apr}''_{\mathbb{C}}(A)$, then there exists a $K_0 \in \mathbb{C}$ such that $x \in K_0 \cap A \neq \emptyset$. By the assumption, there exists $L_0 \in \mathbb{C}'$ such that $K_0 \subseteq L_0$, so $x \in L_0$ and $L_0 \cap A \neq \emptyset$. Hence, $x \in \overline{apr}''_{\mathbb{C}'}(A)$. □

Table 3
Neighborhood systems, minimal and maximal descriptions, and neighborhood operators for Example 1.

x	$\mathscr{C}(\mathbb{C}, x)$	$md(\mathbb{C}, x)$	$MD(\mathbb{C}, x)$	$N_1(x)$	$N_2(x)$	$N_3(x)$	$N_4(x)$
1	{1,123}	{1}	{123}	1	1	123	123
2	{23,123}	{23}	{123}	23	23	123	123
3	{23,123,34}	{23,34}	{123,34}	3	234	3	1234
4	{34}	{34}	{34}	4	34	4	34

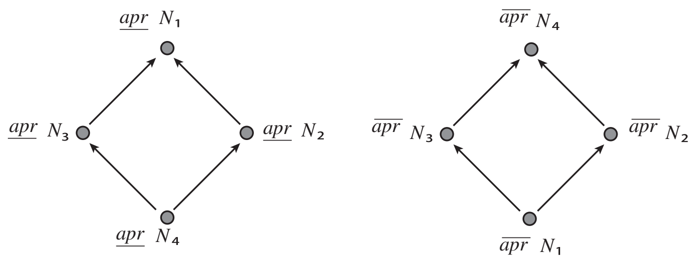

Fig. 1. Partial order for element based approximation operators.

Proposition 14. *Let $\mathbb{C}$ be a covering of U. It holds that:*

$$\overline{apr}''_{\mathbb{C}_3} \leqslant \overline{apr}''_{\mathbb{C}_1} \leqslant \overline{apr}''_{\mathbb{C}_2} = \overline{apr}''_{\mathbb{C}_\cap} \leqslant \overline{apr}''_{\mathbb{C}_4}.$$

Proof. Clearly, $\cap md(\mathbb{C}, x) \subseteq K$ for each $K \in md(\mathbb{C}, x)$ and for all $K \in md(\mathbb{C}, x)$ there exists $L \in MD(\mathbb{C}, x)$ such that $K \subseteq L$ and finally for all $L \in MD(\mathbb{C}, x)$ we have $L \subseteq MD(\mathbb{C}, x)$. Therefore, the result follows as a consequence of Propositions 2 and 13. □

Next, we will relate the lower approximation operators $\underline{apr}'$ and $\underline{apr}''$. First, the following proposition follows easily from the definitions of granule based approximation operators.

Proposition 15. *Let $\mathbb{C}$ be a covering of U. It holds that $\underline{apr}''_{\mathbb{C}} \leqslant \underline{apr}'_{\mathbb{C}}$.*

Proof. Direct from Eqs. (13) and (16). □

Apart from this, we also have the following result.

Proposition 16. $\underline{apr}''_{\mathbb{C}_2} \leqslant \underline{apr}'_{\mathbb{C}_4}$.

Proof. Let $x \in \underline{apr}''_{\mathbb{C}_2}(A)$. Clearly, for all $K \in MD(\mathbb{C}, x)$, it holds that $x \in K$. Therefore, $K \subseteq A$ for all $K \in MD(\mathbb{C}, x)$. From this follows that $\cup MD(\mathbb{C}, x) \subseteq A$ and since $\cup MD(\mathbb{C}, x) \in \mathbb{C}_4$, therefore $x \in \underline{apr}'_{\mathbb{C}_4}(A)$. □

The remaining covering based lower approximation operators are not comparable, as the following examples show.

Example 2. For the covering $\mathbb{C} = \{1, 3, 13, 24, 34, 14, 234\}$ of $U = 1234$, we have:

1. $\mathbb{C}_1 = \{1, 3, 24, 14, 34\}$.
2. $\mathbb{C}_2 = \{13, 14, 234\}$.
3. $\mathbb{C}_3 = \{1, 24, 3, 4\}$.
4. $\mathbb{C}_4 = \{134, 234, 1234\}$.
5. $\mathbb{C}_\cap = \{13, 24, 34, 14, 234\}$.

The lower approximations of all non-empty subsets of U are shown in Table 4. From these results, we can conclude that the pairs $\underline{apr}''_{\mathbb{C}_1} - \underline{apr}'_{\mathbb{C}_4}$, $\underline{apr}''_{\mathbb{C}_1} - \underline{apr}'_{\mathbb{C}_2}$, $\underline{apr}''_{\mathbb{C}_3} - \underline{apr}'_{\mathbb{C}_2}$, $\underline{apr}''_{\mathbb{C}_3} - \underline{apr}'_{\mathbb{C}_4}$ and $\underline{apr}''_{\mathbb{C}_3} - \underline{apr}'_{\mathbb{C}_\cap}$ are not comparable. This example does not allow us to conclude anything about the incomparability of $\underline{apr}''_{\mathbb{C}_3} - \underline{apr}'_{\mathbb{C}_1}$, neither about $\underline{apr}''_{\mathbb{C}_1} - \underline{apr}'_{\mathbb{C}_\cap}$.

Example 3. For the covering $\mathbb{C} = \{1, 12, 123, 24, 23, 234\}$ of $U = 1234$, we have that $\mathbb{C}_1 = \{1, 12, 23, 24\}$ and $\mathbb{C}_\cap = \{1, 12, 123, 24, 234\}$. We can see that: $\underline{apr}'_{\mathbb{C}_\cap}(12) = 12 \supset 1 = \underline{apr}''_{\mathbb{C}_1}(12)$, while $\underline{apr}'_{\mathbb{C}_\cap}(23) = \emptyset \subset 3 = \underline{apr}''_{\mathbb{C}_1}(23)$. Therefore, $\underline{apr}'_{\mathbb{C}_\cap}$ and $\underline{apr}''_{\mathbb{C}_1}$ are not comparable.

Example 4. For the covering $\mathbb{C} = \{13, 14, 23, 24, 34, 234\}$ of $U = 1234$, we have that $\mathbb{C}_1 = \{13, 14, 23, 24, 34\}, \mathbb{C}_3 = \{1, 2, 3, 4\}$. We can see that: $\underline{apr}'_{\mathbb{C}_1}(12) = \emptyset \subset 12 = \underline{apr}''_{\mathbb{C}_3}(12)$. On the other hand, in Example 2, we have $\underline{apr}'_{\mathbb{C}_1}(14) = 14 \supset 1 = \underline{apr}''_{\mathbb{C}_3}(14)$. Therefore, $\underline{apr}''_{\mathbb{C}_3}$ and $\underline{apr}'_{\mathbb{C}_1}$ are not comparable.

Order relations for upper approximation operators $\overline{apr}'_{\mathbb{C}}$ and $\overline{apr}''_{\mathbb{C}}$ can be established as a consequence of duality.

To conclude this subsection, the partial order relations for granule based approximation operators are shown in Fig. 2.

4.3. Partial order for system based definitions

The following example shows that the two system-based lower approximation operators defined in Eqs. (19) and (20) are not comparable. By duality, therefore, the corresponding upper approximations are not comparable, either.

Example 5. Consider the covering $\mathbb{C} = \{2, 12, 23, 14, 124\}$. The corresponding $S_\cap$ and $S_\cup$ can be obtained from the following closure systems and their duals.

- $\cap$-closure $= \{\emptyset, U, 2, 12, 23, 14, 124, 1\}$.
- $\cup$-closure $= \{\emptyset, U, 2, 12, 23, 14, 124, 123\}$.

The lower approximations of all non-empty subsets of U are shown in Table 5. From these values, we can see that $\underline{apr}_{S_\cap}$ and $\underline{apr}_{S_\cup}$ are not comparable.

4.4. Partial order for the operators from the non-dual framework

First, we note that order relations among the first six upper approximation operators $H_1^{\mathbb{C}} - H_6^{\mathbb{C}}$ have already been established in [16].

Proposition 17 [16].

a. $H_1^{\mathbb{C}} \leqslant H_4^{\mathbb{C}} \leqslant H_2^{\mathbb{C}}$.
b. $H_1^{\mathbb{C}} \leqslant H_3^{\mathbb{C}} \leqslant H_2^{\mathbb{C}}$.
c. $H_5^{\mathbb{C}} \leqslant H_1^{\mathbb{C}}$.
d. $H_6^{\mathbb{C}} \leqslant H_2^{\mathbb{C}}$.

In this section, we complete the partial order for this framework, considering also $H_7^{\mathbb{C}}$, by means of the following propositions and example.

Proposition 18. $H_6^{\mathbb{C}} \leqslant H_7^{\mathbb{C}}$.

Proof. Direct from their definitions. □

Proposition 19. $H_7^{\mathbb{C}} \leqslant H_2^{\mathbb{C}}$.

Proof. If $x \in H_7^{\mathbb{C}}(A)$ then $x \in N_1(w)$ for some $w \in U$ and $N_1(w) \cap A \neq \emptyset$. But $N_1(w) \subseteq K \in \mathbb{C}$ for some K, therefore $x \in H_2^{\mathbb{C}}(A)$. □

Proposition 20. $H_5^{\mathbb{C}} \leqslant H_7^{\mathbb{C}}$.

Proof. If $x \in H_5^{\mathbb{C}}(A)$ then $x \in N_1(w)$ for some $w \in A$. Since $w \in N_1(w), N_1(w) \cap A \neq \emptyset$ and $x \in H_7^{\mathbb{C}}(A)$. □

Example 6. Let us consider the covering $\mathbb{C} = \{1, 12, 34, 123\}$ of $U = 1234$. The upper approximations $H_i^{\mathbb{C}}(A)$ of all non-empty subsets A of U are shown in Table 6. From these results, we can see that the pairs $H_7^{\mathbb{C}} - H_1^{\mathbb{C}}$, $H_7^{\mathbb{C}} - H_3^{\mathbb{C}}$ and $H_7^{\mathbb{C}} - H_4^{\mathbb{C}}$ are not comparable.

To summarize, the order relations for the upper approximation operators $H_i^{\mathbb{C}}$ are shown in Fig. 3. Again, by Proposition 6, we can consider the reverse ordering for the lower approximation operators $(H_i^{\mathbb{C}})^\partial$.

Table 4
Granule based lower approximations for Example 2.

A	$\underline{apr}'_{\mathbb{C}_1}$	$\underline{apr}'_{\mathbb{C}_2}$	$\underline{apr}'_{\mathbb{C}_3}$	$\underline{apr}'_{\mathbb{C}_4}$	$\underline{apr}'_{\mathbb{C}_\cap}$	$\underline{apr}''_{\mathbb{C}_1}$	$\underline{apr}''_{\mathbb{C}_2}$	$\underline{apr}''_{\mathbb{C}_3}$	$\underline{apr}''_{\mathbb{C}_4}$
1	1	∅	1	∅	∅	∅	∅	1	∅
2	∅	∅	∅	∅	∅	∅	∅	∅	∅
3	3	∅	3	∅	∅	∅	∅	3	∅
4	∅	∅	4	∅	∅	∅	∅	∅	∅
12	1	∅	1	∅	∅	∅	∅	1	∅
13	13	13	13	∅	13	∅	∅	13	∅
14	14	14	14	∅	14	1	∅	1	∅
23	3	∅	3	∅	∅	∅	∅	3	∅
24	24	∅	24	∅	24	2	∅	24	∅
34	34	∅	34	∅	34	3	∅	3	∅
123	13	13	13	∅	13	∅	∅	13	∅
124	124	14	124	∅	124	12	∅	124	∅
134	134	134	134	134	134	13	1	13	∅
234	234	234	234	234	234	23	2	234	∅
1234	1234	1234	1234	1234	1234	1234	1234	1234	1234

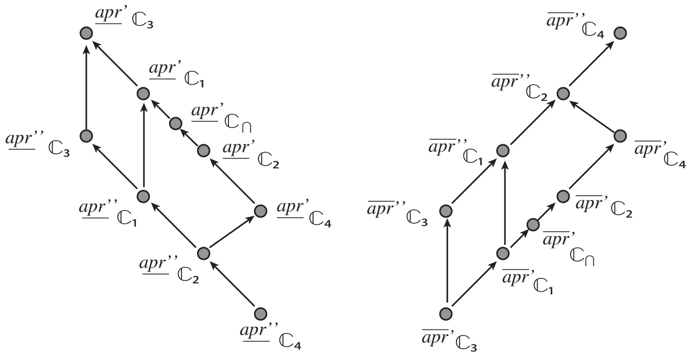

Fig. 2. Partial order for granule based approximation operators.

Table 5
System based lower approximations for Example 5.

A	$\underline{apr}_{S_\cup}$	$\underline{apr}_{S_\cap}$
1	$\emptyset$	$\emptyset$
2	$\emptyset$	2
3	3	$\emptyset$
4	$\emptyset$	$\emptyset$
12	$\emptyset$	12
13	3	$\emptyset$
14	14	14
23	23	23
24	$\emptyset$	2
34	34	$\emptyset$
123	23	123
124	14	124
134	134	14
234	234	23
1234	1234	1234

4.5. Partial order for all approximation operators

To establish the partial order relation among all lower approximation operators considered in Table 1, we first prove the following proposition which relates operators in different groups.

Proposition 21.

a. $\underline{apr}'_{\mathbb{C}_2} \leqslant \underline{apr}_{N_3}$.
b. $\underline{apr}_{N_4} \leqslant \underline{apr}'_{\mathbb{C}_4}$.
c. $\underline{apr}_{N_4} \leqslant \underline{apr}'_{\mathbb{C}_2}$.
d. $\underline{apr}_{N_2} \leqslant \underline{apr}'_{\mathbb{C}_1}$.
e. $\underline{apr}''_{\mathbb{C}_1} \leqslant H_3^{\mathbb{C}}$.
f. $\underline{apr}''_{\mathbb{C}_1} \leqslant \underline{apr}_{N_2}$.
g. $\underline{apr}_{S_\cap} \leqslant (H_5^{\mathbb{C}})^{\partial}$.

Proof.

a. if $x \in \underline{apr}'_{\mathbb{C}_2}(A), x \in K$ for some $K \in \mathbb{C}_2$, and $K \subseteq A$. $N_3(x) = \cap MD(\mathbb{C}, x) \subseteq K \subseteq A$, hence $x \in \underline{apr}_{N_3}(A)$.
b. If $x \in \underline{apr}_{N_4}(A), N_4(x) \subseteq A$. But $N_4(x) \in \mathbb{C}_4$, and $x \in N_4(x)$, hence $x \in \underline{apr}'_{\mathbb{C}_4}(A)$.
c. If $x \in \underline{apr}_{N_4}(A), N_4(x) \subseteq A$. Therefore, for all $K \in \mathbb{C}_2$ with $x \in K$, we have $K \subseteq \cup MD(\mathbb{C}, x) = N_4(x) \subseteq A$, hence $x \in \underline{apr}'_{\mathbb{C}_2}(A)$.
d. If $x \in \underline{apr}_{N_2}(A), N_2(x) = \cup md(\mathbb{C}, x) \subseteq A$. Therefore, for all $K \in \mathbb{C}_1$ with $x \in K$, we have $K \subseteq \cup md(\mathbb{C}, x) = N_2(x) \subseteq A$, hence $x \in \underline{apr}'_{\mathbb{C}_1}(A)$.
e. Remark that, for $A \in \mathscr{P}(U), \underline{apr}''_{\mathbb{C}_1}(A) = \{x \in U : \forall K \in \{md(\mathbb{C}, y) : y \in U\}(x \in K \Rightarrow K \subseteq A)\}$. If $x \in \underline{apr}''_{\mathbb{C}_1}(A)$, then $x \in A$. Since $x \in K$ for all K in $md(\mathbb{C}, x)$, it holds that $x \in H_3^{\mathbb{C}}(A)$.

Table 6
Upper approximations $H_i^{\mathbb{C}}(A)$ for Example 6.

A	$H_1^{\mathbb{C}}$	$H_2^{\mathbb{C}}$	$H_3^{\mathbb{C}}$	$H_4^{\mathbb{C}}$	$H_5^{\mathbb{C}}$	$H_6^{\mathbb{C}}$	$H_7^{\mathbb{C}}$
1	1	123	1	1	1	12	12
2	12	123	12	123	12	2	12
3	1234	1234	1234	1234	3	34	34
4	34	34	34	34	34	4	34
12	12	123	12	12	12	12	12
13	1234	1234	1234	1234	13	1234	1234
14	134	1234	134	134	134	124	1234
23	1234	1234	1234	1234	123	234	1234
24	1234	1234	1234	1234	1234	24	1234
34	34	1234	1234	34	34	34	34
123	123	1234	1234	123	123	1234	1234
124	1234	1234	1234	1234	1234	124	1234
134	134	1234	1234	134	134	1234	1234
234	1234	1234	1234	1234	1234	234	1234
1234	1234	1234	1234	1234	1234	1234	1234

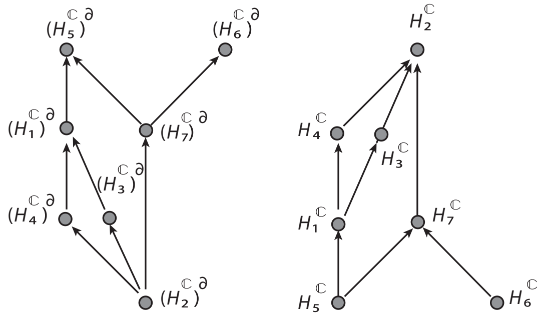

Fig. 3. Partial order relation for lower and upper approximations of the non-dual framework.

f. We will show that $\overline{apr}_{N_2} \leqslant \overline{apr}''_{\mathbb{C}_1}$. If $x \in \overline{apr}_{N_2}(A)$, then $N_2(x) \cap A \neq \emptyset$, but $N_2(x) = \cup\{K : K \in md(\mathbb{C}, x)\}$, so $\cup\{K : K \in md(\mathbb{C}, x)\} \cap A = \cup\{K \cap A : K \in md(\mathbb{C}, x)\} \neq \emptyset$. Thus there exists $K_0 \in md(\mathbb{C}, x)$ such that $K_0 \cap A \neq \emptyset$. Therefore $x \in \cup\{K \in \mathbb{C}_1 : K \cap A \neq \emptyset\} = \overline{apr}''_{\mathbb{C}_1}(A)$ and so, $\overline{apr}_{N_2} \leqslant \overline{apr}''_{\mathbb{C}_1}$. The result [f.] is a consequence of duality.

g. We will see that $H_5^{\mathbb{C}} \leqslant \overline{apr}_{S_\cap}$. For this, let us suppose $w \in H_5^{\mathbb{C}}(A)$, then $w \in N_1(x)$ for some $x \in A$. From Proposition 1.a, we have $N_1(w) \subseteq N_1(x)$. We will show that $w \in X$, for all $X \in (\cap\text{-closure }(\mathbb{C}))$ with $X \supseteq A$. Let X be a set in $(\cap\text{-closure }(\mathbb{C}))$ with $X \supseteq A$, then $x \in A \subseteq X$ and $X = K_1 \cap \ldots \cap K_l$ with $K_j \in \mathbb{C}$. So $x \in K_j$ for all $j = 1, 2, 3, \ldots, l$. Again, from Proposition 1.b, each K_j can be expressed as $K_j = \cup_{x_j \in K_j} N_1(x_j)$, therefore $x \in N_1(x_{j_0})$, for some $x_{j_0} \in K_j$ and $N_1(x) \subseteq K_j$ for all $j = 1, 2, 3, \ldots, l$. Thus we have $w \in N_1(w) \subseteq N_1(x) \subseteq X$. This shows that $w \in \overline{apr}_{S_\cap}$ and that $H_5^{\mathbb{C}}(A) \leqslant \overline{apr}_{S_\cap}(A)$. □

Next, the following examples show that the operator $\underline{apr}_{S_\cap}$ is not comparable with any of the other ones. This is partially a consequence of the fact that when $\mathbb{C}$ is a partition, $\underline{apr}_{S_\cap}$ does not coincide with Pawlak's lower approximation operator.

Example 7. Since the covering $\mathbb{C} = \{1, 2, 3, 4\}$ of $U = 1234$ is a partition, we have that $\underline{apr}(A) = A$, for all $A \subseteq U$ with $\underline{apr}$ any of the lower approximation operators in Table 1 different from $\underline{apr}_{S_\cap}$. On the other hand, $\cap$-closure $(\mathbb{C}) = \{\emptyset, 1234, 1, 2, 3, 4\}$ and $(\cap\text{-closure }(\mathbb{C}))' = \{\emptyset, 1234, 234, 134, 124, 123\}$, so $\underline{apr}_{S_\cap}(A) = \emptyset$, if $|A| < 3$ and $\underline{apr}_{S_\cap}(A) = A$, if $|A| \geqslant 3$. Then $\underline{apr} \nleqslant \underline{apr}_{S_\cap}$.

Example 8. Consider the covering $\mathbb{C} = \{1, 12, 123, 24, 23, 234\}$ of $U = 1234$ in Example 3.

The lower approximations of all non-empty subsets of U for the approximation operators $\underline{apr}_{S_\cap}, \underline{apr}_{N_1}$ and $(H_1^{\mathbb{C}})^{\partial}$ are shown in Table 7. From these values, we can see that $\underline{apr}_{S_\cap}$ is comparable with neither $\underline{apr}_{N_1}$ nor $(H_1^{\mathbb{C}})^{\partial}$.

From Examples 7 and 8, and the Fig. 4 below, we can see that the operator $\underline{apr}_{S_\cap}$ is not comparable with any of the other ones, different from $(H_5^{\mathbb{C}})^{\partial}$.

An integrated Hasse diagram of the partial order relation among the different lower approximation operators can be seen in Fig. 4. A completely analogous diagram can be constructed for the upper approximation operators. We represent each group of operators in Table 1 with a circled number. The green circles represent operators which form an adjoint pair, with their dual; the yellow circles represent meet morphisms which do not form an adjoint pair with their dual; and the red

Table 7
System based lower approximations for Example 8.

A	$\underline{apr}_{S_\cap}$	$\underline{apr}_{N_1}$	$\left(H_1^{\mathbb{C}}\right)^\partial$
1	1	1	1
2	$\emptyset$	2	$\emptyset$
3	$\emptyset$	$\emptyset$	3
4	4	$\emptyset$	4
12	1	12	1
13	13	1	13
14	14	1	14
23	$\emptyset$	23	3
24	4	24	4
34	34	$\emptyset$	34
123	13	123	13
124	14	124	14
134	134	1	$\emptyset$
234	234	234	234
1234	1234	1234	1234

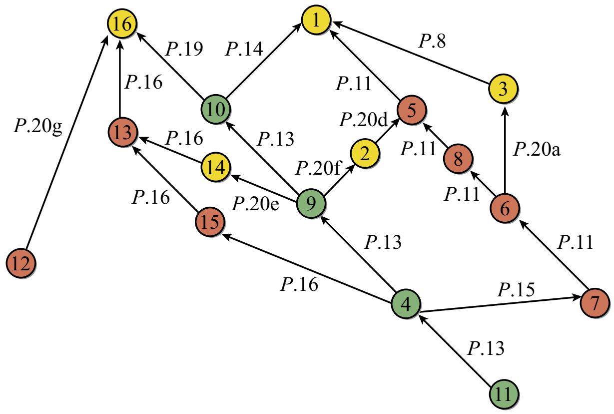

Fig. 4. Partial order relation for groups of lower approximation operators in Table 1.

Table 8
Minimal descriptions, and neighborhood operator N_2 for the coverings $\mathbb{C}_2$ and $\mathbb{C}_\cap$.

x	$md(\mathbb{C}_2, x)$	$md(\mathbb{C}_\cap, x)$	$N_2^{\mathbb{C}_2}(x)$	$N_2^{\mathbb{C}_\cap}(x)$
1	{13,14}	{13,14}	134	134
2	{234}	{24}	234	24
3	{13,234}	{13,34}	1234	134
4	{14,234}	{14,24,34}	1234	1234

circles represent operators which do not form an adjoint pair with their dual, neither are meet morphisms. The label $P.n$ is the number of proposition where the order relation is established.

We can see that there are two top elements: the group (16), represented by $(H_5^{\mathbb{C}})^\partial$ and the group (1), represented by $\underline{apr}_{N_1}$. The unique bottom element is the group (11) represented by $\underline{apr}''_{\mathbb{C}_4}$. Recall that the top elements represent those lower approximation operators for which the quality of classification (24) is highest.

It is interesting to note that while both top elements of the partial order are meet-morphisms, they do not form an adjoint pair with their duals. This can be seen as a disadvantage of these operators, because the adjointness property guarantees for a dual pair $(\underline{apr}, \overline{apr})$ that the fix points of $\underline{apr}$ and $\overline{apr}$ coincide, in other words, $\underline{apr}(A) = A$ iff $\overline{apr}(A) = A$.

If we consider only the subset of lower approximation operators that satisfy adjointness, we can see that they form a chain, with group (10), represented by $\underline{apr}''_{\mathbb{C}_3}$ as the top element.

Finally, the Hasse diagram also suggests some additional approximation operators to be considered. For example, the order relation between the groups (10)–(1), (9)–(5), (4)–(8), (4)–(6) and (11)–(7) corresponds to the relation: $\underline{apr}'_{\mathbb{C}} \leqslant \underline{apr}''_{\mathbb{C}}$. The group (2) is between (9) and (5) and it is defined from $\underline{apr}_{N_2}$. If we consider the neighborhood operators

$N_2^{\mathbb{C}}(x) = \cup\{K : K \in md(\mathbb{C}, x)\}$ for different coverings, we obtain new lower approximation operators: $\underline{apr}_{N_2^{\mathbb{C}_1}}, \underline{apr}_{N_2^{\mathbb{C}_2}}$, $\underline{apr}_{N_2^{\mathbb{C}_3}}, \underline{apr}_{N_2^{\mathbb{C}_4}}$ and $\underline{apr}_{N_2^{\mathbb{C}_\cap}}$. Following the proof in Proposition 21d and f, order relations with the new operators can easily be established:

1. $\underline{apr}''_{\mathbb{C}_3} \leqslant \underline{apr}_{N_2^{\mathbb{C}_3}} \leqslant \underline{apr}'_{\mathbb{C}_3}$.
2. $\underline{apr}''_{\mathbb{C}_1} \leqslant \underline{apr}_{N_2^{\mathbb{C}_1}} \leqslant \underline{apr}'_{\mathbb{C}_1}$.
3. $\underline{apr}''_{\mathbb{C}_2} \leqslant \underline{apr}_{N_2^{\mathbb{C}_\cap}} \leqslant \underline{apr}'_{\mathbb{C}_\cap}$.
4. $\underline{apr}''_{\mathbb{C}_2} \leqslant \underline{apr}_{N_2^{\mathbb{C}_2}} \leqslant \underline{apr}'_{\mathbb{C}_2}$.
5. $\underline{apr}''_{\mathbb{C}_4} \leqslant \underline{apr}_{N_2^{\mathbb{C}_4}} \leqslant \underline{apr}'_{\mathbb{C}_4}$.

Example 9 below shows that some of these new approximations operators are different. In particular we will see that $\underline{apr}''_{\mathbb{C}_2} \neq \underline{apr}_{N_2^{\mathbb{C}_\cap}} \neq \underline{apr}'_{\mathbb{C}_\cap}$. Similar results can be established for other coverings.

Example 9. From the covering $\mathbb{C}$ in Example 2, we have: $\mathbb{C}_2 = \{13, 14, 234\}$ and $\mathbb{C}_\cap = \{13, 24, 34, 14, 234\}$. The minimal description $md(\mathbb{C}, x)$ for these coverings and the neighborhood operators $N_2^{\mathbb{C}}$, are shown in Table 8.

From the neighborhoods in Table 8, the lower approximations of $A = 24$ are: $\underline{apr}_{N_2^{\mathbb{C}_2}}(A) = \emptyset$ and $\underline{apr}_{N_2^{\mathbb{C}_\cap}}(A) = 2$, so $\underline{apr}_{N_2^{\mathbb{C}_2}} \neq \underline{apr}_{N_2^{\mathbb{C}_\cap}}$. Also, we can see that $\underline{apr}''_{\mathbb{C}_2}(A) = \emptyset, \underline{apr}_{N_2^{\mathbb{C}_\cap}}(A) = 2$ and $\underline{apr}'_{\mathbb{C}_\cap}(A) = 24$, therefore $\underline{apr}''_{\mathbb{C}_2} \neq \underline{apr}_{N_2^{\mathbb{C}_\cap}} \neq \underline{apr}'_{\mathbb{C}_\cap}$.

In general, additional approximation operators can be defined combining the different coverings with the neighborhood based lower approximation operators as well as with $(H_i^{\mathbb{C}})^{\partial}$ $(i = 1, \ldots, 7)$. All of them may be included in the Hasse diagram in Fig. 4, but in order not to complicate the visual representation of the partial order, we refrain from doing so here.

5. Conclusions

In this paper, we have studied the order relation between lower and upper approximation operators proposed in the literature for covering-based rough sets. Among the sixteen dual pairs that we have considered in our study, we have identified $\left((H_5^{\mathbb{C}})^{\partial}, H_5^{\mathbb{C}}\right)$ and $(\underline{apr}_{N_1}, \overline{apr}_{N_1}) = \left(\underline{apr}'_{\mathbb{C}_3}, \overline{apr}'_{\mathbb{C}_3}\right)$ as the ones that produce the finest approximations. If additionally adjointness is required, then the finest pair is $\left(\underline{apr}''_{\mathbb{C}_3}, \overline{apr}''_{\mathbb{C}_3}\right)$. These results may guide practitioners who are faced with an ample collection of approximation operators to choose from.

As part of our future work, we would like to obtain further characterizations of covering-based approximation operators. In particular, an interesting question is whether there exist dual pairs of approximation operators that are both idempotent and adjoint. Also, we plan to study different order relations among pairs of approximation operators, for example the extension of the entropy based order relation defined by Zhu and Wen in [30].

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