Duality, Conjugacy and Adjointness of Approximation Operators in Covering Based Rough Sets

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Abstract

Many different proposals exist for the definition of lower and upper approximation operators in covering-based rough sets. In this paper, we establish relationships between the most commonly used operators, using especially concepts of duality, conjugacy and adjointness (also referred to as Galois connection). We highlight the importance of the adjointness condition as a way to provide a meaningful link, aside from duality, between a pair of approximation operators. Moreover, we show that a pair of a lower and an upper approximation operator can be dual and adjoint at the same time if and only if the upper approximation is self conjugate, and we relate this result to a similar characterization obtained for the generalized rough set model based on a binary relation.

Key words: rough sets, coverings, approximations, adjointness, Galois connections.

1. Introduction

Rough set theory, proposed by Pawlak [16] in 1982, is a prominent tool for dealing with uncertainty and incompleteness in information systems. It revolves around the notion of discernibility of objects, which classically is represented by means of an equivalence relation, or equivalently a partition of the set of objects. The theory has been generalized from different perspectives. One generalization of rough sets is to replace the equivalence relation by a general binary relation. In this case, the binary relation determines collections of sets that no longer form a partition of U. This generalization has been used in applications with incomplete information systems and tables with continuous attributes [9, 10, 27]. A second generalization is to replace the partition obtained by the equivalence relation with a covering; i.e., a collection of nonempty sets with union equal to U. There are many works in these two directions and some connections between the two generalizations have been established, for example in [27, 42, 41, 38].

In our work, we focus on covering-based approaches. Recently, Yao and Yao [32] introduced a general framework for the study of dual pairs of covering-based approximation operators, distinguishing between element based, granule based and subsystem based definitions. They emphasized the crucial importance of duality of the lower and upper approximations, stating that without this condition one has no reason to investigate them jointly as a pair. Nevertheless, a lot of other approximation pairs outside this framework have been studied in literature; for instance, in [25], Yang and Li present a summary of seven non dual pairs of approximation operators used by Żakowski [33], Pomykala [17], Tsang et al. [18], Zhu [41], Zhu and Wang [43], Xu and Wang [23]. On the other hand, in a recent paper, Ciucci et al. [4] study opposition diagrams for dual and non-dual approximations in rough set theory, indicating the relevance of both.

In this paper, we draw attention to another important property of classical rough sets, namely the fact that the lower approximation $\underline{apr}$ and the upper approximation $\overline{apr}$ form a Galois connection, or put differently, that they make up an

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adjoint pair. Galois connections have been used in various areas of mathematics and theoretical computer science as a way of providing a strong link between (morphisms on) ordered structures [1, 15]. In the context of rough sets, the adjointness property guarantees for example that the fix points of $\underline{apr}$ and $\overline{apr}$ coincide, in other words, $\underline{apr}(A) = A$ iff $\overline{apr}(A) = A$. This is particularly important when considering the notion of definable sets, i.e., subsets of the universe for which the approximations equal the original set. Moreover, if an upper approximation $\overline{apr}$ has an adjoint $\underline{apr}$, then it is unique, and vice versa a lower approximation $\underline{apr}$ can have at most one co-adjoint, so the Galois connection, like duality, provides a one-to-one connection between approximation operators.

The adjointness property also plays an important role in the definition of logical systems derived from rough sets, in particular modal logics, since lower approximation is related with the necessity operator and upper approximation with the possibility operator. For example, Järvinen et al. describe in [12] an information logic for rough sets using the Galois connection, while Yao shows in [30] that adjointness is necessary for modal logic type B. Furthermore, a Galois Connection also holds between the set theoretic operators used in Formal Concepts Analysis, which bears a close relationship to rough set theory [28].

The objective of this paper is twofold: first, we want to establish relationships between the various definitions of approximation operators in the covering-based rough set model; for this, we will be using some lattice concepts for rough sets, such as meet and join preserving functions, conjugates, duals and Galois connections. Secondly, we want to evaluate the existing proposals with respect to the adjointness condition, providing in particular a characterization of approximation operators pairs that are both dual and adjoint. In this way, we hope to provide a clear cut roadmap for the covering-based rough set landscape, pinpointing the most useful operators among the many that have been proposed in the literature and guiding future research directions. In particular, we believe our results are helpful to select appropriate approximation operators in typical rough set applications such as attribute selection and classification; unlike in the classical case, there are many possibilities to define the approximations, and the particular choice is likely to affect the quality of these applications.

The remainder of this paper is organized as follows. Section 2 presents preliminary concepts about rough sets and lower and upper approximations in covering based rough sets, as well as the necessary lattice concepts about duality, conjugacy and adjointness. In Section 3, we present equivalences and relationships between various approximation operators, evaluate which of them satisfy adjointness, and end with the characterization theorem for dual and adjoint pairs. We also relate this characterization with previous results in generalized rough set theory based on a binary relation. Finally, Section 4 presents some conclusions and outlines future work.

2. Preliminaries

Throughout this paper, we assume that U is a finite and nonempty set. $\mathscr{P}(U)$ represents the collection of subsets of U.

2.1. Three definitions of Pawlak rough sets approximations

In Pawlak's rough set model, an approximation space is an ordered pair $apr = (U, E)$, where E is an equivalence relation on U. According to [29, 32], there are three different, but equivalent ways to define lower and upper approximation operators: element based definition, granule based definition and subsystem based definition. For each $A \subseteq U$, the lower and upper approximations are defined by:

Element based definition.

$$\underline{apr}(A) = \{x \in U : [x]_E \subseteq A\} \tag{1}$$

$$\overline{apr}(A) = \{x \in U : [x]_E \cap A \neq \emptyset\} \tag{2}$$

Granule based definition.

$$\underline{apr}(A) = \bigcup\{[x]_E \in U/E : [x]_E \subseteq A\} \tag{3}$$

$$\overline{apr}(A) = \bigcup\{[x]_E \in U/E : [x]_E \cap A \neq \emptyset\} \tag{4}$$

Subsystem based definition.

$$\underline{apr}(A) = \bigcup\{X \in \sigma(U/E) : X \subseteq A\} \tag{5}$$

$$\overline{apr}(A) = \bigcap\{X \in \sigma(U/E) : X \supseteq A\} \tag{6}$$

where $\sigma(U/E)$ is the σ algebra that is obtained from the equivalence classes U/E, by adding the empty set and making it closed under set unions.

2.2. Covering based rough sets

Many authors have investigated generalized rough set models obtained by changing the condition that E is an equivalence relation, or equivalently, that U/E forms a partition. In particular, we consider the case where the partition is replaced by a covering of U. Covering based rough sets and tolerance relation based rough sets are used in information systems with missing data.

Definition 1. *[36] Let $\mathbb{C} = \{K_i\}$ be a family of nonempty subsets of U. $\mathbb{C}$ is called a covering of U if $\cup K_i = U$. The ordered pair $(U, \mathbb{C})$ is called a covering approximation space.*

It is clear that a partition generated by an equivalence relation is a special case of a covering of U, so the concept of covering is a generalization of a partition.

2.2.1. Framework of dual approximation operators

In [32], Yao and Yao proposed a general framework for the study of covering based rough sets. It is based on the observation that when the partition U/E is generalized to a covering, the different definitions of lower and upper approximations in Section 2.1 are no longer equivalent. A distinguishing characteristic of their framework is the requirement that the obtained lower and upper approximation operators form a dual pair, that is, for $A \subseteq U$, $\underline{apr}(\sim A) = \sim \overline{apr}(A)$, where $\sim A$ represents the complement of A, i.e., $\sim A = U \setminus A$.

Below, we briefly review the generalizations of the element, granule and subsystem based definitions. In the element based definition, equivalence classes are replaced by neighborhood operators:

Definition 2. *[32] A neighborhood operator is a mapping $N : U \to \mathscr{P}(U)$. If $N(x) \neq \emptyset$ for all $x \in U$, N is called a serial neighborhood operator. If $x \in N(x)$ for all $x \in U$, N is called a reflexive neighborhood operator.*

Each neighborhood operator defines an ordered pair $(\overline{apr}_N, \underline{apr}_N)$ of dual approximation operators:

$$\underline{apr}_N(A) = \{x \in U : N(x) \subseteq A\} \tag{7}$$

$$\overline{apr}_N(A) = \{x \in U : N(x) \cap A \neq \emptyset\} \tag{8}$$

Different neighborhood operators, and hence different element based definitions of covering based rough sets, can be obtained from a covering $\mathbb{C}$. In general, we are interested in the sets K in $\mathbb{C}$ such that $x \in K$:

Definition 3. *[32] If $\mathbb{C}$ is a covering of U and $x \in U$, a neighborhood system $\mathscr{C}(\mathbb{C}, x)$ is defined by:*

$$\mathscr{C}(\mathbb{C}, x) = \{K \in \mathbb{C} : x \in K\} \tag{9}$$

In a neighborhood system $\mathscr{C}(\mathbb{C}, x)$, the minimal and maximal sets that contain an element $x \in U$ are particularly important.

Definition 4. *Let $(U, \mathbb{C})$ be a covering approximation space and x in U. The set*

$$md(\mathbb{C}, x) = \{K \in \mathscr{C}(\mathbb{C}, x) : (\forall S \in \mathscr{C}(\mathbb{C}, x), S \subseteq K) \Rightarrow K = S)\} \tag{10}$$

is called the minimal description of x. [2] On the other hand, the set

$$MD(\mathbb{C}, x) = \{K \in \mathscr{C}(\mathbb{C}, x) : (\forall S \in \mathscr{C}(\mathbb{C}, x), S \supseteq K) \Rightarrow K = S\} \tag{11}$$

is called the maximal description of x. [44]

The sets $md(\mathbb{C},x)$ and $MD(\mathbb{C},x)$ represent extreme points of $\mathscr{C}(\mathbb{C},x)$: for any $K \in \mathscr{C}(\mathbb{C},x)$, we can find neighborhoods $K_1 \in md(\mathbb{C},x)$ and $K_2 \in MD(\mathbb{C},x)$ such that $K_1 \subseteq K \subseteq K_2$. From $md(\mathbb{C},x)$ and $MD(\mathbb{C},x)$, Yao and Yao [32] defined the following neighborhood operators:

1. $N_1(x) = \cap\{K : K \in md(\mathbb{C},x)\}$
2. $N_2(x) = \cup\{K : K \in md(\mathbb{C},x)\}$
3. $N_3(x) = \cap\{K : K \in MD(\mathbb{C},x)\}$
4. $N_4(x) = \cup\{K : K \in MD(\mathbb{C},x)\}$

The set $N_1(x) = \cap md(\mathbb{C},x)$ for each $x \in U$, is called the minimal neighborhood of x, and it satisfies some important properties as is shown in the following proposition:

Proposition 1. *[25] Let $\mathbb{C}$ be a covering of U and $K \in \mathbb{C}$, then*

- $K = \cup_{x \in K} N_1(x)$
- *If $y \in N_1(x)$ then $N_1(y) \subseteq N_1(x)$.*

Example 1. *For simplicity, we use a special notation for sets and collections. For example, the set $\{1,2,3\}$ will be denoted by 123 and the collection $\{\{1,2,3\},\{2,3,5\}\}$ will be written as $\{123, 235\}$. Let us consider the covering $\mathbb{C} = \{1,5,6,14,16,123,456,2345,12346,235,23456,2356,12345\}$ of $U = 123456$. The neighborhood system $\mathscr{C}(\mathbb{C},x)$, the minimal description $md(\mathbb{C},x)$ and the maximal description $MD(\mathbb{C},x)$ are listed in Table 1.*

x	$\mathscr{C}(\mathbb{C},x)$	$md(\mathbb{C},x)$	$MD(\mathbb{C},x)$
1	{1, 14, 16, 123, 12346, 12345}	{1}	{12346, 12345}
2	{123, 2345, 12346, 235, 23456, 2356, 12345}	{123, 235}	{12346, 12345, 23456}
3	{123, 2345, 12346, 235, 23456, 2356, 12345}	{123, 235}	{12346, 12345, 23456}
4	{14, 456, 2345, 12346, 23456, 12345}	{14, 456, 2345}	{12346, 12345, 23456}
5	{5, 456, 2345, 235, 2356, 12345}	{5}	{23456, 12345}
6	{6, 16, 456, 12346, 23456, 2356}	{6}	{12346, 23456}

Table 1: Illustration of neighborhood system, minimal and maximal description.

The four neighborhood operators obtained from $\mathscr{C}(\mathbb{C},x)$ are listed in Table 2.

x	$N_1(x)$	$N_2(x)$	$N_3(x)$	$N_4(x)$
1	1	1	1234	123456
2	23	1235	234	123456
3	23	1235	234	123456
4	4	123456	234	123456
5	5	5	2345	123456
6	6	6	2346	123456

Table 2: Illustration of neighborhood operators.

For the set $A = 246$, we have that $\underline{apr}_{N_1}(A) = 46$, because $N_1(x) \subseteq A$ only for $x = 4$ and $x = 6$. $\underline{apr}_{N_2}(A) = 6$ and $\underline{apr}_{N_3}(A) = \underline{apr}_{N_4}(A) = \emptyset$. The upper approximations are: $\overline{apr}_{N_1}(A) = \overline{apr}_{N_2}(A) = 2346$, and $\overline{apr}_{N_3}(A) = \overline{apr}_{N_4}(A) = 123456$.

Generalizing the granule based definitions (3) and (4), the following dual pairs of approximation operators based on a covering $\mathbb{C}$ were considered in [32]:

$$
\begin{aligned}
\underline{apr}'_{\mathbb{C}}(A) &= \bigcup\{K \in \mathbb{C} : K \subseteq A\} = \{x \in U : (\exists K \in \mathbb{C})(x \in K \wedge K \subseteq A)\} && (12)\\
\overline{apr}'_{\mathbb{C}}(A) &= \sim \underline{apr}'_{\mathbb{C}}(\sim A) = \{x \in U : (\forall K \in \mathbb{C})(x \in K \Rightarrow K \cap A \neq \emptyset)\} && (13)\\
\underline{apr}''_{\mathbb{C}}(A) &= \sim \overline{apr}''_{\mathbb{C}}(\sim A) = \{x \in U : (\forall K \in \mathbb{C})(x \in K \Rightarrow K \subseteq A)\} && (14)\\
\overline{apr}''_{\mathbb{C}}(A) &= \bigcup\{K \in \mathbb{C} : K \cap A \neq \emptyset\} = \{x \in U : (\exists K \in \mathbb{C})(x \in K \wedge K \cap A \neq \emptyset)\} && (15)
\end{aligned}
$$

$(\overline{apr}'_{\mathbb{C}}, \underline{apr}'_{\mathbb{C}})$ and $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}})$ are referred to as the tight and loose pair of approximations[1], reflecting the fact that $\underline{apr}''_{\mathbb{C}}(A) \subseteq \underline{apr}'_{\mathbb{C}}(A) \subseteq A \subseteq \overline{apr}'_{\mathbb{C}}(A) \subseteq \overline{apr}''_{\mathbb{C}}(A)$, for all $A \subseteq U$ [6, 17].

Furthermore, Yao and Yao introduced four new coverings derived from a covering $\mathbb{C}$ in [32]:

1. $\mathbb{C}_1 = \cup\{md(\mathbb{C},x) : x \in U\}$
2. $\mathbb{C}_2 = \cup\{MD(\mathbb{C},x) : x \in U\}$
3. $\mathbb{C}_3 = \{\cap(md(\mathbb{C},x)) : x \in U\} = \{\cap(\mathscr{C}(\mathbb{C},x)) : x \in U\}$
4. $\mathbb{C}_4 = \{\cup(MD(\mathbb{C},x)) : x \in U\} = \{\cup(\mathscr{C}(\mathbb{C},x)) : x \in U\}$

For example, the covering $\mathbb{C}_1$ is the collection of all sets in the minimal description of each $x \in U$, while $\mathbb{C}_3$ is the collection of the intersections of minimal descriptions for each $x \in U$, i.e., $\{N_1(x) : x \in U\}$. Additionally, they considered the so-called intersection reduct $\mathbb{C}_\cap$ and union reduct $\mathbb{C}_\cup$ of a covering $\mathbb{C}$:

$$
\begin{aligned}
\mathbb{C}_\cap &= \mathbb{C} \setminus \left\{K \in \mathbb{C} : (\exists \mathbb{K} \subseteq \mathbb{C} \setminus \{K\}) \left(K = \bigcap \mathbb{K}\right)\right\} && (16)\\
\mathbb{C}_\cup &= \mathbb{C} \setminus \left\{K \in \mathbb{C} : (\exists \mathbb{K} \subseteq \mathbb{C} \setminus \{K\}) \left(K = \bigcup \mathbb{K}\right)\right\} && (17)
\end{aligned}
$$

These reducts eliminate intersection (respectively, union) reducible elements from the covering, and it can be proven that they also form a covering of U.

Each of the above six coverings determines two pairs of dual approximations given by equations (12) and (13) and equations (14) and (15), respectively.

Example 2. *The six coverings obtained from the covering $\mathbb{C}$ in Example 1 are:*

1. $\mathbb{C}_1 = \{1, 123, 235, 14, 456, 2345, 5, 6\}$
2. $\mathbb{C}_2 = \{12346, 12345, 23456\}$
3. $\mathbb{C}_3 = \{1, 23, 4, 5, 6\}$
4. $\mathbb{C}_4 = \{123456\}$
5. $\mathbb{C}_\cup = \{1, 123, 235, 14, 456, 2345, 5, 6\}$
6. $\mathbb{C}_\cap = \{123, 456, 2345, 12346, 5, 23456, 14, 16, 2356, 12345\}$

The lower and upper approximations of $A = 246$ using the operators discussed above are shown in Table 3.

Covering	$\underline{apr}''_{\mathbb{C}}$	$\underline{apr}'_{\mathbb{C}}$	$\overline{apr}'_{\mathbb{C}}$	$\overline{apr}''_{\mathbb{C}}$
$\mathbb{C}$	$\emptyset$	6	2346	123456
$\mathbb{C}_1$	$\emptyset$	6	2346	12346
$\mathbb{C}_2$	$\emptyset$	$\emptyset$	123456	123456
$\mathbb{C}_3$	46	46	2346	2346
$\mathbb{C}_4$	$\emptyset$	123456	123456	123456
$\mathbb{C}_\cup$	$\emptyset$	6	2346	123456
$\mathbb{C}_\cap$	$\emptyset$	$\emptyset$	12346	123456

Table 3: Illustration of granule-based definitions of approximation operations.

[1] By contrast, note that in [5], $\overline{apr}'_{\mathbb{C}}$ is called tight upper approximation, referring to the fact that all elements of $\mathscr{C}(\mathbb{C},x)$ are taken into account, giving rise to a strict or "tight" requirement; on the other hand, $\underline{apr}'_{\mathbb{C}}$ is called loose lower approximation in the same paper, meaning that we only look at the best element in $\mathscr{C}(\mathbb{C},x)$, which is a more flexible, "loose" demand.

Finally, to generalize the subsystem based definitions (5) and (6), Yao and Yao used the notion of a closure system over U, i.e., a family of subsets of U that contains U and is closed under set intersection. Given a closure system $\mathbb{S}$ over U, one can construct its dual system $\mathbb{S}'$, containing the complements of each K in $\mathbb{S}$, as follows:

$$\mathbb{S}' = \{\sim K : K \in \mathbb{S}\} \tag{18}$$

The system $\mathbb{S}'$ contains $\emptyset$ and it is closed under set union. Given $S = (\mathbb{S}', \mathbb{S})$, a pair of dual lower and upper approximations can be defined as follows:

$$\underline{apr}_S(A) = \bigcup\{K \in \mathbb{S}' : K \subseteq A\} \tag{19}$$

$$\overline{apr}_S(A) = \bigcap\{K \in \mathbb{S} : K \supseteq A\} \tag{20}$$

As a particular example of a closure system, [32] considered the so-called intersection closure $S_{\cap,\mathbb{C}}$ of a covering $\mathbb{C}$, i.e., the minimal subset of $\mathscr{P}(U)$ that contains $\mathbb{C}$, $\emptyset$ and U, and is closed under set intersection. On the other hand, the union closure of $\mathbb{C}$, denoted by $S_{\cup,\mathbb{C}}$, is the minimal subset of $\mathscr{P}(U)$ that contains $\mathbb{C}$, $\emptyset$ and U, and is closed under set union. It can be shown that the dual system $S'_{\cup,\mathbb{C}}$ forms a closure system. Both $S_\cap = ((S_{\cap,\mathbb{C}})', S_{\cap,\mathbb{C}})$ and $S_\cup = (S_{\cup,\mathbb{C}}, (S_{\cup,\mathbb{C}})')$ can be used to obtain pairs of dual approximation operations by means of Eqs. (19) and (20).

Example 3. *For the covering $\mathbb{C}$, the intersection and union closure can be obtained as follows.*

- $S_{\cap,\mathbb{C}} = \mathbb{C} \cup \{\emptyset, 1, 24, 5, 13, 56, 6, 123456\}$
- $S_{\cup,\mathbb{C}} = \mathbb{C} \cup \{\emptyset, 1236, 1235, 1234, 12356, 1456, 156, 146, 15, 145, 56, 123456\}$

The corresponding lower approximations of $A = 246$ are: $\underline{apr}_{S_\cap}(A) = 6$ and $\underline{apr}_{S_\cup}(A) = 6$. The upper approximations are: $\overline{apr}_{S_\cap}(A) = 2346$ and $\overline{apr}_{S_\cup}(A) = 2346$.

Summarizing, twenty pairs of dual approximation operators were defined in this framework: four from the element based definition based on neighborhood operators, fourteen from the granule based definition, based on the covering $\mathbb{C}$ and six derived coverings, and two from the subsystem based definition, using intersection and union closure. All pairs are listed in Table 4.

#	Dual pair		#	Dual pair	
1	$\overline{apr}_{N_1}$	$\underline{apr}_{N_1}$	2	$\overline{apr}_{N_2}$	$\underline{apr}_{N_2}$
3	$\overline{apr}_{N_3}$	$\underline{apr}_{N_3}$	4	$\overline{apr}_{N_4}$	$\underline{apr}_{N_4}$
5	$\overline{apr}'_{\mathbb{C}}$	$\underline{apr}'_{\mathbb{C}}$	6	$\overline{apr}''_{\mathbb{C}}$	$\underline{apr}''_{\mathbb{C}}$
7	$\overline{apr}'_{\mathbb{C}_1}$	$\underline{apr}'_{\mathbb{C}_1}$	8	$\overline{apr}''_{\mathbb{C}_1}$	$\underline{apr}''_{\mathbb{C}_1}$
9	$\overline{apr}'_{\mathbb{C}_2}$	$\underline{apr}'_{\mathbb{C}_2}$	10	$\overline{apr}''_{\mathbb{C}_2}$	$\underline{apr}''_{\mathbb{C}_2}$
11	$\overline{apr}'_{\mathbb{C}_3}$	$\underline{apr}'_{\mathbb{C}_3}$	12	$\overline{apr}''_{\mathbb{C}_3}$	$\underline{apr}''_{\mathbb{C}_3}$
13	$\overline{apr}'_{\mathbb{C}_4}$	$\underline{apr}'_{\mathbb{C}_4}$	14	$\overline{apr}''_{\mathbb{C}_4}$	$\underline{apr}''_{\mathbb{C}_4}$
15	$\overline{apr}'_{\mathbb{C}_\cap}$	$\underline{apr}'_{\mathbb{C}_\cap}$	16	$\overline{apr}''_{\mathbb{C}_\cap}$	$\underline{apr}''_{\mathbb{C}_\cap}$
17	$\overline{apr}'_{\mathbb{C}_\cup}$	$\underline{apr}'_{\mathbb{C}_\cup}$	18	$\overline{apr}''_{\mathbb{C}_\cup}$	$\underline{apr}''_{\mathbb{C}_\cup}$
19	$\overline{apr}_{S_\cap}$	$\underline{apr}_{S_\cap}$	20	$\overline{apr}_{S_\cup}$	$\underline{apr}_{S_\cup}$

Table 4: List of dual pairs of approximation operators considered in [32].

2.2.2. Framework of non-dual approximation operators

Another important line of research on covering-based rough sets has focused on pairs of approximation operators that are not necessarily dual. The first of such proposals goes back to Żakowski [33], who was in fact the first to generalize Pawlak's rough set theory from a partition to a covering. In recent papers [19, 25], a total of two lower approximation operators and seven upper approximation are summarized, which are listed below.[2]

[2] There is no uniform notation for these approximation operators in literature. For example in [34], *SH* refers to the sixth upper approximation, while in [40] *SH* refers to the second upper approximation. For ease of reference, here we use numerical subscripts in the definitions.

For a covering $\mathbb{C}$ of U, the principal lower approximations for $A \subseteq U$ are:

- $L_1^{\mathbb{C}}(A) = \cup\{K \in \mathbb{C} : K \subseteq A\} = \underline{apr}'_{\mathbb{C}}(A)$
- $L_2^{\mathbb{C}}(A) = \cup\{N_1(x) : x \in U \wedge N_1(x) \subseteq A\}$

It can be checked that $L_2^{\mathbb{C}}$ is the particular case of $L_1^{\mathbb{C}}$ when we use $\mathbb{C}_3$ instead of $\mathbb{C}$, so $L_2^{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_3}$. The seven upper approximations are listed as follows:

- $H_1^{\mathbb{C}}(A) = L_1^{\mathbb{C}}(A) \cup (\cup\{md(\mathbb{C},x) : x \in A - L_1^{\mathbb{C}}(A)\})$
- $H_2^{\mathbb{C}}(A) = \cup\{K \in \mathbb{C} : K \cap A \neq \emptyset\} = \overline{apr}''_{\mathbb{C}}(A)$
- $H_3^{\mathbb{C}}(A) = \cup\{md(\mathbb{C},x) : x \in A\}$
- $H_4^{\mathbb{C}}(A) = L_1^{\mathbb{C}}(A) \cup (\cup\{K : K \cap (A - L_1^{\mathbb{C}}(A)) \neq \emptyset\})$
- $H_5^{\mathbb{C}}(A) = \cup\{N_1(x) : x \in A\}$
- $H_6^{\mathbb{C}}(A) = \{x : N_1(x) \cap A \neq \emptyset\} = \overline{apr}_{N_1}(A)$
- $H_7^{\mathbb{C}}(A) = \cup\{N_1(x) : N_1(x) \cap A \neq \emptyset\}$

The couple $(H_1^{\mathbb{C}}, L_1^{\mathbb{C}})$ was proposed by Żakowski in [33]. Pomykala [17] considered $(H_2^{\mathbb{C}}, L_1^{\mathbb{C}})$, while Tsang et al. studied $(H_3^{\mathbb{C}}, L_1^{\mathbb{C}})$ in [18]. Zhu and Wang defined $(H_4^{\mathbb{C}}, L_1^{\mathbb{C}})$ in [43], while $(H_5^{\mathbb{C}}, L_1^{\mathbb{C}})$ was considered by Zhu in [41]. Xu and Wang proposed $(H_6^{\mathbb{C}}, L_2^{\mathbb{C}})$, and finally $(H_7^{\mathbb{C}}, L_2^{\mathbb{C}})$ was discussed by Xu and Zhang in [24].

The definitions of $L_2^{\mathbb{C}}$, $H_5^{\mathbb{C}}$, $H_6^{\mathbb{C}}$ and $H_7^{\mathbb{C}}$ can generate other approximations if N_1 is replaced by another neighborhood operator. The operators $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$, $H_4^{\mathbb{C}}$ and $H_5^{\mathbb{C}}$ do not appear explicitly in Yao's framework, although $H_1^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$ can be expressed as union of a lower and an upper approximation operator. For example, $H_1^{\mathbb{C}}$ can be expressed as:

$$H_1^{\mathbb{C}}(A) = \underline{apr}'_{\mathbb{C}}(A) \cup \overline{apr}''_{\mathbb{C}}(A - \underline{apr}'_{\mathbb{C}}(A)) \tag{21}$$

Example 4. *In Table 5, we present the upper approximations for some subsets of U and the covering $\mathbb{C} = \{12, 124, 25, 256, 345, 26, 6\}$. Since there are no two identical columns, we can conclude that all seven upper approximations are different.*

A	$H_1^{\mathbb{C}}$	$H_2^{\mathbb{C}}$	$H_3^{\mathbb{C}}$	$H_4^{\mathbb{C}}$	$H_5^{\mathbb{C}}$	$H_6^{\mathbb{C}}$	$H_7^{\mathbb{C}}$
16	126	12456	126	1246	126	16	126
25	25	123456	123456	25	25	1235	12345
123	12345	123456	123456	12345	12345	123	12345
34	12345	12345	12345	12345	345	34	345

Table 5: Illustration of upper aproximations $H_1^{\mathbb{C}}$–$H_7^{\mathbb{C}}$.

2.3. Dual, conjugate and adjoint operators

In this subsection, we recall the basic relations among dual, conjugate and adjoint operators, following the ideas introduced by Järvinen in [11] in the context of lattices.

Let L be a bounded lattice with a least element 0 and greatest element 1. For $a \in L$, we say that $b \in L$ is a complement of a if $a \vee b = 1$ and $a \wedge b = 0$. A distributive and bounded lattice with complement for all $a \in L$ is called a Boolean lattice. In particular, the collection $\mathscr{P}(U)$ of subsets of a set U, with least element $\emptyset$, greatest element U and intersection, union and complement operations is a Boolean lattice.

2.3.1. Meet and join morphisms

If L,K are lattices, a map $f : L \to K$ is a complete join morphism if whenever $S \subseteq L$ and $\vee S$ exists in L, then $\vee f(S)$ exists in K and $f(\vee S) = \vee f(S)$. Analogously, a map $f : L \to K$ is a complete meet morphism if whenever $S \subseteq L$ and $\wedge S$ exists in L, then $\wedge f(S)$ exists in K and $f(\wedge S) = \wedge f(S)$.

A finite lattice is always complete, i.e. $\vee S$ and $\wedge S$ exist for all $S \subseteq L$. In this case a meet morphism f (a morphism that satisfies $f(a \wedge b) = f(a) \wedge f(b)$ for a and b in L) is a complete meet morphism, and dually, a join morphism f (a morphism that satisfies $f(a \vee b) = f(a) \vee f(b)$ for a and b in L) is a complete join morphism. Since in this paper, we assume that U is a finite universe, for the approximation operators we consider it will thus be sufficient to establish that they are meet (resp., join) morphisms.

We recall some known results about the approximation operators from Section 2.2.2. Zhu showed that $H_2^{\mathbb{C}}$, $H_3^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$ are join morphisms in [37, 40, 44], while Wu et al. in [21] showed this for $H_5^{\mathbb{C}}$.

Moreover, in [36] it was shown that the upper approximation $H_1^{\mathbb{C}}$ is a join morphism and the lower approximation $L_1^{\mathbb{C}}$ is a meet morphism if and only if $\mathbb{C}$ is a unary covering. Recall that a covering $\mathbb{C}$ is unary if for all $x \in U$, $md(\mathbb{C}, x)$ is a singleton, or equivalently if $\forall K_1, K_2 \in \mathbb{C}$, $K_1 \cap K_2$ is a union of elements of $\mathbb{C}$. [44] As a particular example, the covering $\mathbb{C}_3$ obtained from any covering $\mathbb{C}$ is a unary covering.

2.3.2. Duality

Definition 5. *[11] Let $f, g : B \to B$ be two self-maps on a complete Boolean lattice B. We say that g is the dual of f, if for all $x \in B$,*

$$g(\sim x) = \sim f(x),$$

where $\sim x$ represents the complement of $x \in B$.

For any f, we denote by f^{∂} the dual of f. If $g = f^{\partial}$ then $f = g^{\partial}$.

An interesting relation between meet morphisms, join morphisms and duality can be seen in the following proposition:

Proposition 2. *[20] If $(\overline{apr}, \underline{apr})$ is a dual pair of approximation operators, then $\underline{apr}$ is a meet morphism if and only if $\overline{apr}$ is a join morphism.*

2.3.3. Conjugacy

Definition 6. *[11] Let f and g be two self-maps on a complete Boolean lattice B. We say that g is a conjugate of f, if for all $x, y \in B$,*

$$x \wedge f(y) = 0 \text{ if and only if } y \wedge g(x) = 0.$$

If g is a conjugate of f, then f is a conjugate of g. If a map f is the conjugate of itself, then f is called self-conjugate. The conjugate of f will be denoted as f^c.

Proposition 3. *[11] Let f be a self-map on a complete Boolean lattice B. Then f has a conjugate if and only if f is a complete join morphism on B.*

From the proof given in [11], the conjugate of f is defined, for $y \in B$, by:

$$g(y) = \sim (\vee \{x : f(x) \leq \sim y\}) = \wedge \{\sim x : f(x) \wedge y = 0\} \tag{22}$$

In the context of rough sets, the concept of conjugate is related to upper approximation operators. Here, we also define a dual notion of co-conjugate for lower approximations, as follows:

Definition 7. *Let f and g be two self-maps on a complete Boolean lattice B. We say that g is a co-conjugate of f, if for all $x, y \in B$,*

$$x \vee f(y) = 1 \text{ if and only if } y \vee g(x) = 1.$$

If g is a co-conjugate of f, then f is a co-conjugate of g. If a map f is the co-conjugate of itself, then f is called self co-conjugate. The co-conjugate of f will be denoted as f_c.

Proposition 4. *If (f_1,g_1) and (f_2,g_2) are pairs of dual self-maps on a complete Boolean lattice B, then f_1 and f_2 are conjugate if and only if g_1 and g_2 are co-conjugate.*

PROOF. *We show just one part of the equivalence. The other part is similar.*
If $f_1^c = f_2$ then, for $y \in B$:

$$\begin{aligned} x \vee g_1(y) = 1 &\Leftrightarrow \sim x \wedge \sim g_1(y) = 0 \\ &\Leftrightarrow \sim x \wedge f_1(\sim y) = 0 \\ &\Leftrightarrow \sim y \wedge f_2(\sim x) = 0 \\ &\Leftrightarrow y \vee \sim f_2(\sim x) = 1 \\ &\Leftrightarrow y \vee g_2(x) = 1 \end{aligned}$$

So, $(g_1)_c = g_2$. □

2.3.4. Adjointness

The idea of adjoint can be found in various settings in mathematics and theoretical computer science. We consider the particular case of adjoints defined on preordered sets, known as Galois connection.

Definition 8. *[11] Let P and Q be two preorders; an ordered pair (f,g) of maps $f : P \to Q$ and $g : Q \to P$ is called a Galois connection if for all $p \in P$ and $q \in Q$,*

$$f(p) \leq q \text{ if and only if } p \leq g(q). \tag{23}$$

The map g is called the *adjoint* of f and will be denoted as f^a. The map f is called the *co-adjoint* of g and will be denoted as g_a. It is easy to show that the maps are order-preserving, i.e., if $p \leq p'$ then $f(p) \leq f(p')$ and if $q \leq q'$, then $g(q) \leq g(q')$, and that the adjointness condition (23) is equivalent to the condition that f and g are order-preserving and that for all p and q, $p \leq g(f(p))$ and $f(g(q)) \leq q$.

Conditions about the existence of adjoints and co-adjoints of a morphism between complete lattices are given in the following proposition.

Proposition 5. *[11] Let K and L be complete lattices.*

1. *A map $f : L \to K$ has an adjoint if and only if f is a complete join morphism.*
2. *A map $g : K \to L$ has a co-adjoint if and only if g is a complete meet morphism.*

In this case, the adjoint of f is given by:

$$f^a(y) = \vee\{x \in L : f(x) \leq y\} \tag{24}$$

and the co-adjoint of g is obtained as:

$$g_a(y) = \wedge\{x \in K : y \leq g(x)\}. \tag{25}$$

The following important proposition establishes the relationship between duality, conjugacy and adjointness, and will be used frequently in the next section.

Proposition 6. *[11] Let B be a complete Boolean lattice. For any complete join morphism f on B, its adjoint is the dual of the conjugate of f. On the other hand, for any complete meet morphism g on B, its co-adjoint is the conjugate of the dual of g.*

In classical rough set theory, the lower and upper approximations $(\overline{apr}, \underline{apr})$ form a Galois connection on $\mathscr{P}(U)$. Järvinen shows in [11] that there also exist Galois connections in generalized rough set based on a binary relation. In particular, if R is a binary relation on U and $x \in U$, the sets:

$$R(x) = \{y \in U : xRy\} \text{ and } R^{-1}(x) = \{y \in U : yRx\} \tag{26}$$

are called successor and predecessor neighborhoods, respectively. The ordered pairs $(\overline{apr}_R, \underline{apr}_{R^{-1}})$ and $(\overline{apr}_{R^{-1}}, \underline{apr}_R)$, defined using the element based definitions (7) and (8), with $R(x)$ and $R^{-1}(x)$ instead of $N(x)$, form adjoint pairs. Moreover, $(\overline{apr}_R, \underline{apr}_R)$ and $(\overline{apr}_{R^{-1}}, \underline{apr}_R^{-1})$ are dual pairs.

On the other hand, Yao [26] established the following important proposition which relates dual pairs of approximation operators with the relation-based generalized rough set model considered by Järvinen.

Proposition 7. *[26] Suppose $(\overline{apr}, \underline{apr}) : \mathscr{P}(U) \to \mathscr{P}(U)$ is a dual pair of approximation operators, such that $\overline{apr}$ is a join morphism and $\overline{apr}(\emptyset) = \emptyset$. There exists a symmetric relation R on U, such that $\underline{apr}(A) = \underline{apr}_R(A)$ and $\overline{apr}(A) = \overline{apr}_R(A)$ for all $A \subseteq U$ if and only if the pair $(\overline{apr}, \underline{apr})$ satisfies: $A \subseteq \underline{apr}(\overline{apr}(A))$.*

By duality, we know that $\overline{apr}$ is join morphism if and only if $\underline{apr}$ is a meet morphism and $\overline{apr}(\emptyset) = \emptyset$ if and only if $\underline{apr}(U) = U$. According to the proof, the symmetric relation R is defined by, for x, y in U,

$$xRy \Leftrightarrow x \in \overline{apr}(\{y\}) \tag{27}$$

3. Relationship among approximation operators

In this section, we want to relate the two groups of approximation operators discussed in Sections 2.2.1 and 2.2.2, using the concepts of duality, conjugacy and adjointness. Additionally, we derive a characterization of operators that satisfy both the duality and adjointness condition.

We start the section with a proposition that allows to compute the adjoint of an upper approximation operator in a computationally efficient way. According to Proposition 5, an upper approximation operator $H : \mathscr{P}(U) \to \mathscr{P}(U)$ has an adjoint if and only if H is a join morphism. This adjoint is given by:

$$H^a(A) = \cup\{B \subseteq U : H(B) \subseteq A\}. \tag{28}$$

Hence, the adjoint must be calculated on subsets of U. However, the following proposition provides a less complex alternative.

Proposition 8. *If $H : \mathscr{P}(U) \to \mathscr{P}(U)$ is a join morphism, then the adjoint of H can be calculated by, for $A \subseteq U$:*

$$H^a(A) = \{x \in A : H(\{x\}) \subseteq A\}. \tag{29}$$

PROOF. *We will show that the following equality holds:*

$$\{x \in A : H(\{x\}) \subseteq A\} = \cup\{B \subseteq A : H(B) \subseteq A\}.$$

If $x \in A$ and $H(\{x\}) \subseteq A$ then $\{x\} \subseteq \cup\{B \subseteq A : H(\{x\}) \subseteq A\}$ and $x \in \cup\{B \subseteq A : H(\{x\}) \subseteq A\}$. On the other hand, if $x \in \cup\{B \subseteq A : H(B) \subseteq A\}$, there exists $B = \{y_1, \dots, y_n\}$, such that $H(B) \subseteq A$ and $x \in B$. Because H is a join morphism, $H(B) = \cup_{i=1}^{n} H(\{y_i\})$, from which follows that $H(\{x\}) \subseteq A$. □

This form of the adjoint approximation operator is actually the same as that of the Wybraniec-Skardowska lower approximation operator [22]. Recall that the Wybraniec-Skardowska approximation operator pair $(\overline{apr}_h, \underline{apr}_h)$ is defined as:

$$\underline{apr}_h(A) = \{x \in U : \emptyset \neq h(x) \subseteq A\} \tag{30}$$

$$\overline{apr}_h(A) = \bigcup_{x \in A} h(x) \tag{31}$$

where h is an upper approximation distribution, i.e., an $U \to \mathscr{P}(U)$ mapping that satisfies $\bigcup_{x \in U} h(x) = U$. In this case, for all upper approximations H such that $H(\{x\}) \neq \emptyset$, equations (29) and (30) are the same. h can be considered as a restriction of H to the singletons. The pair of approximation operators given by Eqs. (30) and (31) are not dual operators, but they are an adjoint pair, by definition.

3.1. Non-dual framework of approximation operators

We first establish an important conjugacy relation between the upper approximation operators $H_5^{\mathbb{C}}$ and $H_6^{\mathbb{C}}$. This relationship holds regardless of the neighborhood operator N which is used in the definition, so we begin by proving the following more general proposition.

Proposition 9. *Let N be a neighborhood operator and $G_5^N(A) = \cup\{N(x) : x \in A\}$, $G_6^N(A) = \{x \in U : N(x) \cap A \neq \emptyset\}$ operators defined for N, then G_5^N is the conjugate of G_6^N.*

PROOF. *We show that $A \cap G_5^N(B) \neq \emptyset$ if and only if $B \cap G_6^N(A) \neq \emptyset$, for $A, B \subseteq U$.*

If $A \cap G_5^N(B) \neq \emptyset$, then there exists $w \in U$ such that $w \in A$ and $w \in G_5^N(B)$. Since $w \in G_5^N(B)$, there exists $x_0 \in B$ such that $w \in N(x_0)$. Then $N(x_0) \cap A \neq \emptyset$, with $x_0 \in G_6^N(B)$. Since $x_0 \in B$, then $B \cap G_6^N(A) \neq \emptyset$.

If $B \cap G_6^N(A) \neq \emptyset$, then there exists $w \in U$ such that $w \in B$ and $w \in G_6^N(A)$, i.e., $w \in B$ and $N(w) \cap A \neq \emptyset$. Then there exists z such that $z \in N(w)$ and $z \in A$. Since $z \in N(w)$ and $w \in B$, then $z \in G_5^N(B)$. So, $z \in A \cap G_5^N(B)$, with $A \cap G_5^N(B) \neq \emptyset$. □

Corollary 1. *$H_5^{\mathbb{C}}$ and $H_6^{\mathbb{C}}$ are conjugates.*

PROOF. *In this case, the operators $H_5^{\mathbb{C}}$ and $H_6^{\mathbb{C}}$ correspond to G_5^N and G_6^N, when neighborhood operator N_1 is used.* □

Next, we prove that $L_2^{\mathbb{C}}$ is the adjoint of $H_5^{\mathbb{C}}$. For this, we need the following lemma.

Lemma 1. *For all $w \in U$, $H_5^{\mathbb{C}}(N_1(w)) = N_1(w)$.*

PROOF. *By Proposition 1, from $x \in N_1(w)$ follows $N_1(x) \subseteq N_1(w)$, hence $H_5^{\mathbb{C}}(N_1(w)) \subseteq N_1(w)$. On the other hand, it is clear that $N_1(w) \subseteq H_5^{\mathbb{C}}(N_1(w))$, since $w \in N_1(w)$.* □

Proposition 10. $L_2^{\mathbb{C}} = (H_5^{\mathbb{C}})^a$.

PROOF. *We will show that $L_2^{\mathbb{C}}(A) \subseteq (H_5^{\mathbb{C}})^a(A)$ and $(H_5^{\mathbb{C}})^a(A) \subseteq L_2^{\mathbb{C}}(A)$, for $A \subseteq U$. If $w \in L_2^{\mathbb{C}}(A)$, there exists $x \in U$ such that $w \in N_1(x)$ with $N_1(x) \subseteq A$. The upper approximation $H_5^{\mathbb{C}}$ of $N_1(x)$ is equal to $N_1(x)$, by Lemma 1; i.e., $H_5^{\mathbb{C}}(N_1(x)) = N_1(x)$. Hence, $w \in \cup\{Y \subseteq U : H_5^{\mathbb{C}}(Y) \subseteq A\}$, so $w \in (H_5^{\mathbb{C}})^a(A)$. On the other hand, if $w \in (H_5^{\mathbb{C}})^a(A)$, then there exists $Y \subseteq U$, such that $w \in Y$ and $H_5^{\mathbb{C}}(Y) \subseteq A$; i.e., $\cup\{N_1(x) : x \in Y\} \subseteq A$; in particular, $w \in N_1(w) \subseteq H_5^{\mathbb{C}}(Y) \subseteq A$, so $w \in L_2^{\mathbb{C}}(A)$.* □

Corollary 2. *The dual of $H_6^{\mathbb{C}}$ is equal to $L_2^{\mathbb{C}}$.*

PROOF. According to Propositions 10, 6 and corollary 1, we have: $L_2^{\mathbb{C}} = (H_5^{\mathbb{C}})^a = ((H_5^{\mathbb{C}})^c)^\partial = (H_6^{\mathbb{C}})^\partial$. □

The upper approximation operators $H_2^{\mathbb{C}}$ and $H_7^{\mathbb{C}}$ are closely related; they are discussed in the next two propositions.

Proposition 11. *$H_2^{\mathbb{C}}$ is self-conjugate.*

PROOF. *According to Proposition 6 and the fact that $H_2^{\mathbb{C}}$ is a join morphism, $(H_2^{\mathbb{C}})^a = ((H_2^{\mathbb{C}})^c)^\partial$, so $H_2^{\mathbb{C}}$ is self-conjugate if and only if $(H_2^{\mathbb{C}})^a = (H_2^{\mathbb{C}})^\partial$, that is $(H_2^{\mathbb{C}})^a(\sim A) = \sim H_2^{\mathbb{C}}(A)$. We show that $(H_2^{\mathbb{C}})^a(\sim A) = \sim H_2^{\mathbb{C}}(A)$ for any $A \subseteq U$. $x \notin H_2^{\mathbb{C}}(A)$ if and only if $N_1(x) \cap A = \emptyset$ if and only if $N_1(x) \subseteq \sim A$ if and only if $x \in (H_2^{\mathbb{C}})^a(\sim A)$.* □

Proposition 12. $H_7^{\mathbb{C}} = H_2^{\mathbb{C}_3}$

PROOF. From the definition of $H_7^{\mathbb{C}}$ and $\mathbb{C}_3$, we can see that, for all $A \subseteq U$:

$$\begin{aligned} H_7^{\mathbb{C}}(A) &= \{N_1(x) : N_1(x) \cap A \neq \emptyset\} \\ &= \{K \in \mathbb{C}_3 : K \cap A \neq \emptyset\} \\ &= H_2^{\mathbb{C}_3}(A). \end{aligned}$$

□

Corollary 3. *$H_7^{\mathbb{C}}$ is self-conjugate and its adjoint is equal to $(H_2^{\mathbb{C}_3})^\partial = \underline{apr}''_{\mathbb{C}_3}$.*

PROOF. *Using $H_7^{\mathbb{C}} = H_2^{\mathbb{C}_3}$ and proposition 11, we have that $H_7^{\mathbb{C}}$ is self-conjugate. By proposition 6, we have: $(H_7^{\mathbb{C}})^a = ((H_7^{\mathbb{C}})^c)^\partial = (H_7^{\mathbb{C}})^\partial = (\overline{apr}''_{\mathbb{C}_3})^\partial = \underline{apr}''_{\mathbb{C}_3}$.* □

Finally, we investigate whether any of the remaining upper approximation operators $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$ forms an adjoint pair with $L_1^{\mathbb{C}}$.

Example 5. *In Table 6, we compare lower approximations for some subsets obtained with $L_1^{\mathbb{C}}$, and with the adjoints of $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$. Since none of the final three columns is identical to the first one, we conclude that none of $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$ or $H_4^{\mathbb{C}}$ forms an adjoint pair with $L_1^{\mathbb{C}}$.*

Set	$L_1^{\mathbb{C}}$	$(H_1^{\mathbb{C}})^a$	$(H_3^{\mathbb{C}})^a$	$(H_4^{\mathbb{C}})^a$
246	6	6	64	6
145	145	15	$\emptyset$	15
123	123	1	1	1

Table 6: Comparison of the adjoints of $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$ with $L_1^{\mathbb{C}}$.

From the above results, we can draw the following conclusion: none of the pairs $(H_1^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_2^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_3^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_4^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_5^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_6^{\mathbb{C}}, L_2^{\mathbb{C}})$ and $(H_7^{\mathbb{C}}, L_2^{\mathbb{C}})$ which have previously been considered in the literature (see Section 2.2.2) forms an adjoint pair; on the other hand, $(H_5^{\mathbb{C}}, L_2^{\mathbb{C}})$ does, but this is not a dual pair.

3.2. Dual framework of approximation operators

In this section, we examine the twenty pairs of approximation operators considered by Yao and Yao in [32]. First, we establish an important proposition that follows from the duality of these operators.

3.2.1. Element based definitions

We first establish that upper (resp., lower) approximation element based definitions have adjoints (resp., co-adjoints).

Proposition 13. *For any neighborhood operator N, $\underline{apr}_N$ is a meet morphism.*

PROOF. *Since $\underline{apr}_N(A) = \{x \in U : N(x) \subseteq A\}$, we have $x \in \underline{apr}_N(A \cap B)$ iff $N(x) \subseteq A \cap B$ iff $N(x) \subseteq A$ and $N(x) \subseteq B$ iff $x \in \underline{apr}_N(A)$ and $x \in \underline{apr}_N(B)$ iff $x \in \underline{apr}_N(A) \cap \underline{apr}_N(B)$.* □

Corollary 4. *For any neighborhood operator N, $\overline{apr}_N$ is a join morphism.*

Corollary 5. *For any neighborhood operator N, $\underline{apr}_N$ has a co-adjoint and it is equal to the conjugate of $\overline{apr}_N$.*

PROOF. By Proposition 6 and the duality of $\overline{apr}_N$ and $\underline{apr}_N$, $\left(\underline{apr}_N\right)_a = \left(\underline{apr}_N^\partial\right)^c = (\overline{apr}_N)^c = (G_6^N)^c = G_5^N$. □

Corollary 6. *For any neighborhood operator N, $\overline{apr}_N$ has an adjoint and it is equal to the dual of G_5^N.*

PROOF. Indeed, by Proposition 6, we find $(\overline{apr}_N)^a = ((\overline{apr}_N)^c)^\partial = \left(G_5^N\right)^\partial$.

The remaining question now is whether $(\overline{apr}_N, \underline{apr}_N)$ can ever form an adjoint pair. For this to hold, based on the above we need to have that $\left(\underline{apr}_N\right)_a = G_5^N = G_6^N = \overline{apr}_N$.

Proposition 14. *$(\overline{apr}_N, \underline{apr}_N)$ is an adjoint pair if and only if N satisfies $G_5^N = G_6^N$.*

The following proposition characterizes the neighborhood operators N that satisfy $G_5^N = G_6^N$, and establishes the link with the generalized rough set model based on a binary relation.

Proposition 15. *Let N be a neighborhood operator. The following are equivalent:*

(i) *For all x, y in U, N satisfies*
$$y \in N(x) \Rightarrow x \in N(y) \tag{32}$$

(ii) $G_5^N = G_6^N$

(iii) *There exists a symmetric binary relation R on U such that $N(x) = \{y \in U : xRy\}$.*

PROOF. *We first prove (i) $\Rightarrow$ (ii). Let $A \subseteq U$. If $w \in G_5^N(A)$, then $w \in \cup\{N(x) : x \in A\}$. This means that $w \in N(x)$ for some $x \in A$, and by (32) $x \in N(w)$, so $N(w) \cap A \neq \emptyset$. Hence $w \in G_6^N(A)$.*

If $w \in G_6^N(A)$, then $N(w) \cap A \neq \emptyset$. In other words, there exists $x \in U$ with $x \in A$ and $x \in N(w)$. By (32), $w \in N(x)$ and thus $w \in \cup\{N(x) : x \in A\} = G_5^N(A)$.

On the other hand, to prove (ii) $\Rightarrow$ (i), by the definition of G_5^N, we have $G_5^N(\{x\}) = N(x)$. If $G_5^N(A) = G_6^N(A)$, for all $A \subseteq U$ and $y \in N(x)$ then $N(x) \cap \{y\} \neq \emptyset$, so $x \in G_6^N(\{y\}) = G_5^N(\{y\}) = N(y)$.

Finally, the equivalence (i) $\Leftrightarrow$ (iii) is immediate, with R defined by $xRy \Leftrightarrow x \in N(y)$ for x, y in U.

□

The proposition thus shows that the only adjoint pairs among element-based definitions are those for which the neighborhood is defined by Eq. (26), with symmetric R. The following example shows that for none of the neighborhood operators considered in Section 2.2.1, the adjointness holds.

Example 6. *We illustrate the fact that $(\overline{apr}_{N_i}, \underline{apr}_{N_i})$ is not an adjoint pair for $i = 1, \ldots, 4$, by showing that the property $f(g(x)) \leq x$, satisfied by any Galois connection (f, g), does not hold for them.*

For the covering $\mathbb{C} = \{12, 124, 25, 256, 345, 26, 6\}$ of $U = 123456$, the neighborhoods N_i for the elements of U are shown in Table 7.

x	$N_1(x)$	$N_2(x)$	$N_3(x)$	$N_4(x)$
1	12	12	124	124
2	2	1256	2	12456
3	345	345	345	345
4	4	12345	4	12345
5	5	23456	5	23456
6	6	6	256	256

Table 7: Neighborhood operators for the covering in Example 6.

We have that:

- $\overline{apr}_{N_1}(\underline{apr}_{N_1}(45)) = \overline{apr}_{N_1}(45) = 345 \nsubseteq 45$.
- $\overline{apr}_{N_2}(\underline{apr}_{N_2}(12)) = \overline{apr}_{N_2}(1) = 124 \nsubseteq 12$.
- $\overline{apr}_{N_3}(\underline{apr}_{N_3}(12)) = \overline{apr}_{N_3}(2) = 126 \nsubseteq 2$.
- $\overline{apr}_{N_4}(\underline{apr}_{N_4}(124)) = \overline{apr}_{N_4}(124) = 12345 \nsubseteq 124$.

3.3. Granule based definitions

We first prove some propositions that provide a relationship between different granule based definitions, and between particular element and granule based definitions.

Proposition 16. $\underline{apr}'_{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_1}$.

PROOF. *Let $A \subseteq U$ be a subset of U. For all $K \in \mathbb{C}$ with $K \subseteq A$ there exists a $K' \in md(\mathbb{C},x)$ for some $x \in U$, such that $K' \subseteq K \subseteq A$, so $\cup\{K \in md(\mathbb{C},x) : x \in U\} \subseteq \cup\{K \subseteq \mathbb{C} : K \subseteq A\}$, then $\underline{apr}_{\mathbb{C}_1} \leq \underline{apr}_{\mathbb{C}}$. On the other hand, if $w \in K \subseteq A$, there exists $K' \in md(\mathbb{C},w)$ such that $w \in K' \subseteq K \subseteq A$, so $w \in \cup\{K \in \mathbb{C}_1 : K \subseteq A\}$.* □

Proposition 17. $\underline{apr}'_{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_\cup}$.

PROOF. *We will show that for each $x \in U$, $md(\mathbb{C},x) = md(\mathbb{C}_\cup,x)$ and so, by Proposition 16, we have $\underline{apr}'_{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_\cup}$.*

From the definition of $\mathbb{C}_\cup$, we know that $\mathbb{C}_\cup$ is the $\cup$-reduct and $\mathbb{C}_\cup \subseteq \mathbb{C}$ and $md(\mathbb{C},x) \subseteq md(\mathbb{C}_\cup,x)$. If $K \in md(\mathbb{C}_\cup,x)$ and let us suppose that $K \notin md(\mathbb{C},x)$ there exists K' $md(\mathbb{C},x)$ such that $x \in K' \subseteq K$. □

Proposition 18. $\overline{apr}''_{\mathbb{C}} = \overline{apr}''_{\mathbb{C}_2}$

PROOF. *Clearly, we have $\mathbb{C}_2 \subseteq \mathbb{C}$, and therefore $\overline{apr}''_{\mathbb{C}_2}(A) \subseteq \overline{apr}''_{\mathbb{C}}(A)$, for $A \subseteq U$.*

On the other hand, for each $K \in \mathbb{C}$ there exists $K' \in \mathbb{C}_2$ such that $K \subseteq K'$, so $K \cap A \neq \emptyset$ implies $K' \cap A \neq \emptyset$, thus $\overline{apr}''_{\mathbb{C}}(A) \subseteq \overline{apr}''_{\mathbb{C}_2}(A)$. □

Proposition 19. $\overline{apr}''_{\mathbb{C}} = \overline{apr}''_{\mathbb{C}_\cap}$

PROOF. *From the relation $\mathbb{C}_\cap \subseteq \mathbb{C}$, we have $\overline{apr}''_{\mathbb{C}_\cap}(A) \subseteq \overline{apr}''_{\mathbb{C}}(A)$, for all $A \subseteq U$.*

If $K \in \mathbb{C} - \mathbb{C}_\cap$ there exists $K' \subseteq \mathbb{C} - \{K\}$ such that $K = \bigcap K'$, $K \subseteq L$ for all $L \in K'$, thus if $K \cap A \neq \emptyset$ then $L \cap A \neq \emptyset$, thus $\overline{apr}''_{\mathbb{C}}(A) \subseteq \overline{apr}''_{\mathbb{C}_\cap}(A)$. □

Proposition 20. $\underline{apr}'_{\mathbb{C}_3} = \underline{apr}_{N_1}$.

PROOF.

$$\begin{aligned}\underline{apr}_{N_1}(A) &= \{x \in U : N_1(x) \subseteq A\} \\ &= \cup\{N_1(x) : N_1(x) \subseteq A\} = \underline{apr}'_{\mathbb{C}_3}(A)\end{aligned}$$

For the second equality, if $w \in \underline{apr}'_{\mathbb{C}_3}(A)$, there exists x such that $w \in N_1(x) \subseteq A$. By proposition 1, $N_1(w) \subseteq N_1(x) \subseteq A$, so $w \in \underline{apr}_{N_1}(A)$. □

By Propositions 13 and 20, we have the following corollary.

Corollary 7. *The approximation operator $\underline{apr}'_{\mathbb{C}_3}$ is a meet morphism, but $(\overline{apr}'_{\mathbb{C}_3}, \underline{apr}'_{\mathbb{C}_3})$ is not an adjoint pair.*

In general, the dual pairs $(\overline{apr}'_{\mathbb{C}}, \underline{apr}'_{\mathbb{C}})$ are not adjoint, because we know that $\underline{apr}'_{\mathbb{C}} = L_1^{\mathbb{C}}$ is not a meet morphism when the covering $\mathbb{C}$ is not unary. Next, we study the case of the loose approximation operators $\overline{apr}''_{\mathbb{C}}$ and $\underline{apr}''_{\mathbb{C}}$.

Proposition 21. $\overline{apr}''_{\mathbb{C}}$ *is a self-conjugate join morphism.*

PROOF. *By definition $\overline{apr}''_{\mathbb{C}} = H_2^{\mathbb{C}}$, so it is a join morphism and by Proposition 11 it is self-conjugate.* □

Using Proposition 2, we can establish that $\underline{apr}''_{\mathbb{C}}$ is a meet morphism, which allows us to prove the following result.

Proposition 22. *The pair $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}})$ is an adjoint pair.*

PROOF. *From propositions 6 and 21, we have: $(\overline{apr}''_{\mathbb{C}})^a = [(\overline{apr}''_{\mathbb{C}})^c]^\partial = [\overline{apr}''_{\mathbb{C}}]^\partial = \underline{apr}''_{\mathbb{C}}$ and $(\underline{apr}''_{\mathbb{C}})_a = [(\underline{apr}''_{\mathbb{C}})^\partial]^c = [\overline{apr}''_{\mathbb{C}}]^c = \overline{apr}''_{\mathbb{C}}$.* □

Moreover, the following proposition shows that this adjoint pair can also be seen as a particular case of an element-based definition.

Proposition 23. $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}}) = (\overline{apr}_N, \underline{apr}_N)$*, where N is defined by*

$$N(x) = \{y \in U : (\exists K \in \mathbb{C})(x \in K \wedge y \in K)\} \tag{33}$$

PROOF. *By Proposition 7, we know that there exists a symmetric relation R on U such that $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}}) = (\overline{apr}_R, \underline{apr}_R)$, where $xRy \Leftrightarrow x \in \overline{apr}''_{\mathbb{C}}(\{y\}) \Leftrightarrow x \in \cup\{K \in \mathbb{C} : K \cap \{y\} \neq \emptyset\} \Leftrightarrow x \in \cup\{K \in \mathbb{C} : y \in K\}$.*

Putting $N(x) = R(x)$, we find that $y \in N(x)$ if and only if there exists $K \in \mathbb{C}$ such that $x \in K$ and $y \in K$, or in other words $N(x) = \{y \in U : (\exists K \in \mathbb{C})(x \in K \wedge y \in K)\}$. $\square$

Summarizing, only the loose pair of granule based approximation operators in Yao and Yao's framework is an adjoint pair, and moreover this pair coincides with a particular element-based definition.

To conclude this section, we point out an error in [32]: it is stated there on page 104 that $\underline{apr}''_{\mathbb{C}} = \underline{apr}_{N_4}$ and $\overline{apr}''_{\mathbb{C}} = \overline{apr}_{N_4}$, however this incorrect; in particular, knowing that $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}})$ is an adjoint pair and $(\overline{apr}_{N_4}, \underline{apr}_{N_4})$ is not, this equality cannot hold.

3.4. Subsystem based definitions

First, looking at the definitions we can observe that $\underline{apr}_{S_\cap} = L_1^{S_{\cap,\mathbb{C}}}$ and $\underline{apr}_{S_\cup} = L_1^{(S_{\cup,\mathbb{C}})'}$. Furthermore, we can establish the following relationship between $\underline{apr}_{S_\cup}$ and the granule-based model.

Proposition 24. $\underline{apr}_{S_\cup} = \underline{apr}'_{\mathbb{C}}$

PROOF. *Clearly, $\mathbb{C} \subseteq S_{\cup,\mathbb{C}}$, and therefore $\underline{apr}'_{\mathbb{C}}(A) \subseteq \underline{apr}_{S_\cup}(A)$, for $A \subseteq U$.*

On the other hand, for each $X \in S_{\cup,\mathbb{C}}$, there exists $\mathbb{K} \subseteq \mathbb{C}$ such that $X = \bigcup \mathbb{K}$, thus if $X \subseteq A$ then $L \subseteq A$, for all $L \in \mathbb{K}$. Hence $\cup\{X \in S_{\cup,\mathbb{C}} : X \subseteq A\} \subseteq \cup\{K \in \mathbb{C} : K \subseteq A\}$, i.e., $\underline{apr}_{S_\cup}(A) \subseteq \underline{apr}'_{\mathbb{C}}(A)$. $\square$

This equality can be also be understood by the fact that adding unions of elements to a covering does not refine the covering.

The following example shows that the approximation operators $\underline{apr}_{S_\cap}$ and $\underline{apr}_{S_\cup}$ are not meet morphisms, and neither $\overline{apr}_{S_\cap}$ nor $\overline{apr}_{S_\cup}$ are join morphisms, so they cannot form adjoint pairs.

Example 7. *Consider the subsystems $S_\cap$ and $S_\cup$ from Example 3.*

If $A = 123$, $B = 2456$, then $A \cap B = 2$, $\underline{apr}_{S_\cap}(A) = 123$, $\underline{apr}_{S_\cap}(B) = 2456$ and $\underline{apr}_{S_\cap}(A \cap B) = \emptyset$, then $\underline{apr}_{S_\cap}(A \cap B) \neq \underline{apr}_{S_\cap}(A) \cap \underline{apr}_{S_\cap}(B) = 2$.

On the other hand, If $A = 1236$, $B = 1235$, then $A \cap B = 23$, $\underline{apr}_{S_\cup}(A) = 1236$, $\underline{apr}_{S_\cup}(B) = 1235$ and $\underline{apr}_{S_\cup}(A \cap B) = \emptyset$, then $\underline{apr}_{S_\cup}(A \cap B) \neq \underline{apr}_{S_\cup}(A) \cap \underline{apr}_{S_\cup}(B)$=123.

Analogously, it can be verified that $\overline{apr}_{S_\cap}$ and $\overline{apr}_{S_\cup}$ are not join morphisms.

3.5. Summary of relationships and properties

In Table 8, we summarize the results established in the previous subsections. In particular, we rearrange the group of 20 dual pairs considered by Yao and Yao [32] into 14 groups of equivalent operators, showing in each case their equivalence with members of the non-dual framework considered in [25]. For each group, we also indicate whether the operators form an adjoint pair and whether their members are join/meet morphisms.

For instance, group A consists of the dual pairs 1 and 11 from Table 4 which are equal due to Proposition 20. They are equivalent to the pair $(H_6^{\mathbb{C}}, L_2^{\mathbb{C}})$, because $\underline{apr}_{\mathbb{C}_3} = L_2^{\mathbb{C}}$ by definition, and the dual of $L_2^{\mathbb{C}}$ is $H_6^{\mathbb{C}}$ by Corollary 2. They are join and meet morphisms, but not an adjoint pair as shown in Example 6.

It is interesting to note that all pairs of approximation operators in Yao and Yao's framework can be described from approximation operators $L_1^{\mathbb{C}}$, $L_2^{\mathbb{C}}$, $H_2^{\mathbb{C}}$ and $H_6^{\mathbb{C}}$, their duals and/or their conjugates.

Group	#	Dual pair		Equivalent pair		Adjoint pair	Meet/Join
A	1, 11	$\overline{apr}_{N_1}$ $\overline{apr}_{\mathbb{C}_3}$	$\underline{apr}_{N_1}$ $\underline{apr}_{\mathbb{C}_3}$	$H_6^{\mathbb{C}}$	$L_2^{\mathbb{C}}$	✗	✓
B	2	$\overline{apr}_{N_2}$	$\underline{apr}_{N_2}$	$G_6^{N_2}$	$(G_6^{N_2})^\partial$	✗	✓
C	3	$\overline{apr}_{N_3}$	$\underline{apr}_{N_3}$	$G_6^{N_3}$	$(G_6^{N_3})^\partial$	✗	✓
D	4	$\overline{apr}_{N_4}$	$\underline{apr}_{N_4}$	$G_6^{N_4}$	$(G_6^{N_4})^\partial$	✗	✓
E	5, 7, 17, 20	$\overline{apr}'_{\mathbb{C}}$ $\overline{apr}'_{\mathbb{C}_1}$ $\overline{apr}'_{\mathbb{C}_\cup}$ $\overline{apr}_{S_\cup}$	$\underline{apr}'_{\mathbb{C}}$ $\underline{apr}'_{\mathbb{C}_1}$ $\underline{apr}'_{\mathbb{C}_\cup}$ $\underline{apr}_{S_\cup}$	$(L_1^{\mathbb{C}})^\partial$	$L_1^{\mathbb{C}}$	✗	✗
F	9	$\overline{apr}'_{\mathbb{C}_2}$	$\underline{apr}'_{\mathbb{C}_2}$	$(L_1^{\mathbb{C}_2})^\partial$	$L_1^{\mathbb{C}_2}$	✗	✗
G	13	$\overline{apr}'_{\mathbb{C}_4}$	$\underline{apr}'_{\mathbb{C}_4}$	$(L_1^{\mathbb{C}_4})^\partial$	$L_1^{\mathbb{C}_4}$	✗	✗
H	15	$\overline{apr}'_{\mathbb{C}_\cap}$	$\underline{apr}'_{\mathbb{C}_\cap}$	$(L_1^{\mathbb{C}_\cap})^\partial$	$L_1^{\mathbb{C}_\cap}$	✗	✗
I	6, 10, 18	$\overline{apr}''_{\mathbb{C}}$ $\overline{apr}''_{\mathbb{C}_2}$ $\overline{apr}''_{\mathbb{C}_\cup}$	$\underline{apr}''_{\mathbb{C}}$ $\underline{apr}''_{\mathbb{C}_2}$ $\underline{apr}''_{\mathbb{C}_\cup}$	$H_2^{\mathbb{C}}$	$(H_2^{\mathbb{C}})^\partial$	✓	✓
J	8	$\overline{apr}''_{\mathbb{C}_1}$	$\underline{apr}''_{\mathbb{C}_1}$	$H_2^{\mathbb{C}_1}$	$(H_2^{\mathbb{C}_1})^\partial$	✓	✓
K	12	$\overline{apr}''_{\mathbb{C}_3}$	$\underline{apr}''_{\mathbb{C}_3}$	$H_2^{\mathbb{C}_3}$	$(H_2^{\mathbb{C}_3})^\partial$	✓	✓
L	14	$\overline{apr}''_{\mathbb{C}_4}$	$\underline{apr}''_{\mathbb{C}_4}$	$H_2^{\mathbb{C}_4}$	$(H_2^{\mathbb{C}_4})^\partial$	✓	✓
M	16	$\overline{apr}''_{\mathbb{C}_\cap}$	$\underline{apr}''_{\mathbb{C}_\cap}$	$H_2^{\mathbb{C}_\cap}$	$(H_2^{\mathbb{C}_\cap})^\partial$	✓	✓
N	19	$\overline{apr}_{S_\cap}$	$\underline{apr}_{S_\cap}$	$(L_1^{S_\cap})^\partial$	$L_1^{S_\cap}$	✗	✗

Table 8: Summary of relationships and properties of the approximation operators.

3.6. *Characterization of dual adjoint pairs*

In this section, we characterize pairs of dual and adjoint approximation operators in the covering-based rough set framework.

First, in the left hand side of Figure 1 we illustrate schematically the relations among duality, conjugacy and adjointness. The arrow ∂ represents a dual transformation and the arrows c and co represent transformations of conjugate and co-conjugates, respectively. The pairs (f_i, g_i) are dual, f_1 and f_2 are conjugate and g_1 and g_2 are co-conjugate. The adjoint and co-adjoint can be obtained after two consecutive transformations, so the adjoint of f_1 is g_2, and the co-adjoint of g_1 is f_2. The middle diagram represents the specific situation for the pair $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}})$, while the right hand side represents the case of approximation operators defined from a binary relation as considered by Järvinen [11].

$$\begin{tikzcd}
f_1 \arrow[r, leftrightarrow, "\partial"] \arrow[d, leftrightarrow, "c"'] & g_1 \arrow[d, leftrightarrow, "co"] \\
f_2 \arrow[r, leftrightarrow, "\partial"'] & g_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{apr}''_{\mathbb{C}} \arrow[r, leftrightarrow, "\partial"] \arrow[d, leftrightarrow, "c"'] & \underline{apr}''_{\mathbb{C}} \arrow[d, leftrightarrow, "co"] \\
\overline{apr}''_{\mathbb{C}} \arrow[r, leftrightarrow, "\partial"'] & \underline{apr}''_{\mathbb{C}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{apr}_{R} \arrow[r, leftrightarrow, "\partial"] \arrow[d, leftrightarrow, "c"'] & \underline{apr}_{R} \arrow[d, leftrightarrow, "co"] \\
\overline{apr}_{R^{-1}} \arrow[r, leftrightarrow, "\partial"'] & \underline{apr}_{R^{-1}}
\end{tikzcd}$$

Figure 1: Arrow diagram for approximation operators.

The following important proposition establishes the relationship between duality, conjugacy and adjointness in the general case of a complete Boolean lattice.

Proposition 25. *Let (f,g) be a dual pair on a complete Boolean lattice B. The pair (f,g) is a Galois connection if and only if f is self-conjugate.*

PROOF. *If (f,g) is a Galois connection, then $g = f^a = (f^c)^\partial$. By duality $g = f^\partial$, so $f^a = (f^c)^\partial = f^\partial$, hence $f = f^c$. On the other hand, if $f = f^c$, then $f^a = (f^c)^\partial = (f)^\partial = g$ and $g_a = (g^\partial)^c = f^c = f$.*

In general, if DP represents dual pairs, GC Galois connections and SC pairs for which the upper approximation is self-conjugate, then we have the following implications:

$$GC + SC \rightarrow DP \tag{34}$$

$$DP + SC \rightarrow GC \tag{35}$$

To summarize, Figure 2 contains a set diagram which depicts pairs of approximation operators. **P** is the set of pairs of approximation operators (H,L). **D** contains all the dual pairs and **A** the adjoint pairs. In **D** we have the dual pairs of Yao and Yao's framework. The pairs in the intersection are precisely those pairs of approximations for which the upper approximation is self conjugate. Outside of $\mathbf{D} \cup \mathbf{A}$ there are other pairs which are neither dual nor adjoint, such as $(H_7^{\mathbb{C}}, L_2^{\mathbb{C}})$. The pairs of approximations in Yao and Yao's framework are represented with the letters from A to N and correspond to the groups in Table 8. Finally, the pair $(\overline{apr}_h, \underline{apr}_h)$, defined from an upper approximation distribution h (Wybraniec-Skardowska) is also an adjoint, but not dual pair.

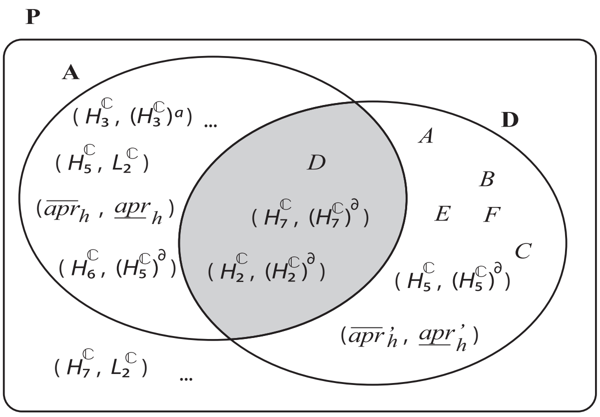

Figure 2: Set diagram of pairs of approximation operators.

4. Conclusion and future work

In this paper, we have studied relationships between pairs of lower and upper approximation operators within the covering-based rough set model. We have shown in particular that within the framework of twenty dual pairs of approximation operators proposed by Yao and Yao in [32], only fourteen of them are different, and of these only five pairs are adjoint. On the other hand, we have demonstrated that none of the pairs of approximation operators $(H_1^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_2^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_3^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_4^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_5^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_6^{\mathbb{C}}, L_2^{\mathbb{C}})$ and $(H_7^{\mathbb{C}}, L_2^{\mathbb{C}})$ considered in e.g. [19, 25] is adjoint; on the other hand, $(H_5^{\mathbb{C}}, L_2^{\mathbb{C}})$ is an adjoint, non-dual pair. Furthermore, we have established that all operators in Yao and Yao's framework can be equivalently expressed in terms of $L_1^{\mathbb{C}}$, $L_2^{\mathbb{C}}$, $H_2^{\mathbb{C}}$ and $H_6^{\mathbb{C}}$.

We have also derived a characterization of dual and adjoint pairs in terms of the self-conjugacy of the upper approximation operator, and have related this equivalence to previous results established for generalized rough sets based on a symmetric binary relation.

As future work, it is interesting to study which are the coverings for which a specific adjoint pair is dual, and conversely, for dual pairs.

On the other hand, a very important continuation of this work involves studying order relationships that hold between the various approximation operators, such as $\overline{apr}''_{\mathbb{C}}(A) \subseteq \overline{apr}'_{\mathbb{C}}(A)$. Such order relations have already been studied partially in [35], where they are induced by means of new entropy and co-entropy measures for covering-based rough sets, and in [14], which studies order relations between various types of neighborhood-related covering-based rough sets. Since approximation operators are used frequently in data analysis applications of rough sets such as feature selection and classification (see e.g. [3] in the case of covering-based rough sets), order relationships can be meaningfully used to guide the selection of appropriate pairs of approximation operators.

The results of this paper may also be applied to Formal Concept Analysis (FCA); FCA and rough set theory have the formal context as a common framework [28]. A formal concept (U,A,R) is defined by two finite sets U (objects) and A (attributes), and a binary relation from U to A. As explained in [13], a regular formal context defines a covering $\mathbb{C}_A$ as the set of object sets of attributes $a \in A$. A dual pair of approximation operators is then derived from this covering, using the operator $\underline{apr}'_{\mathbb{C}}$. However, we may consider other approximation operators associated with a particular covering $\mathbb{C}$, and it makes sense to study their properties in the context of FCA.

As a final important line of future work, we want to extend the obtained results to fuzzy rough set theory, and in particular to the ambit of fuzzy coverings, such as those studied in [7, 8].

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