The Collapsing Method of Defuzzification for Discretised Interval Type-2 Fuzzy Sets

Sarah Greenfield, Francisco Chiclana, Robert John, Simon Coupland
Centre for Computational Intelligence
De Montfort University
Leicester LE1 9BH, UK
{*sarahg, chiclana, rij, simonc*}@*dmu.ac.uk*

Abstract

Defuzzification of type-2 fuzzy sets is a computationally intense problem. This paper proposes a new approach for defuzzification of interval type-2 fuzzy sets. The collapsing method converts an interval type-2 fuzzy set into a *representative embedded set (RES)* which, being a type-1 set, can then be defuzzified straightforwardly. The novel *Representative Embedded Set Theorem (REST)*, with which the method is inextricably linked, is expounded, stated and proved within this paper. Additionally the *Pseudo Representative Embedded Set (PRES)*, a useful and easily calculated approximation to the RES, is discussed.

1 Introduction

1.1 Type-2 Fuzzy Inferencing Systems

Type-2 fuzzy sets were originally proposed by Zadeh [6]; their advantage over type-1 fuzzy sets is their ability to model second-order uncertainties.

A fuzzy inferencing system (FIS) is a computerised aid to decision making, which uses fuzzy sets. It works by applying fuzzy logic operators to common-sense linguistic rules. An FIS can be of any type; here we are concerned with the type-2 FIS, i.e. one which employs type-2 sets. An FIS (of any type) usually starts with a crisp number, and passes through the five stages of fuzzification, antecedent computation, implication, aggregation/combination of consequents, and defuzzification.

When implementing a type-2 FIS researchers discretise the membership functions. This work is concerned only with discretised type-2 fuzzy sets.

1.2 Defuzzification

The output of the fourth stage of a type-2 FIS is a type-2 fuzzy set, which requires *defuzzification* to convert it into a crisp number, the 'answer' to the problem presented to the FIS. It is this final defuzzification stage that is the focus of this paper.

The defuzzification techniques available for discretised type-2 sets are:

Full Defuzzification In a conventional generalised type-2 FIS, the defuzzification stage consists of two parts, type-reduction and defuzzification proper. Type-reduction ([5], pages 248-252) in which the type-2 fuzzy set is converted to a type-1 fuzzy set, the type-reduced set (TRS), involves the processing of *all* the embedded sets ([5], definition 3-10, page 98) within the original discretised type-2 set. These sets are very numerous. For instance, when a prototype type-2 FIS performed an inference using sets which had been discretised into 51 slices across both the *x* and *y*-axes, the number of embedded sets in the aggregated set was calculated to be in the order of 2.9×10^{63}. Though individually easily processed, embedded sets under full defuzzification give rise to a processing bottleneck simply by virtue of their high cardinality.

Sampling Method The sampling method of defuzzification [1] is an efficient, cut-down alternative to full defuzzification. In this technique, only a relatively small random sample of the totality of embedded sets is processed. The resultant defuzzified value, though surprisingly accurate, is nonetheless an approximation.

The Karnik-Mendel Iterative Procedure This procedure [2] is an efficient method of defuzzification for interval type-2 sets but produces only a (very good) approximation to full defuzzification since it works by finding the mid-point of the TRS interval without taking account of the distribution of the values along the interval.

Table 1 contrasts these three methods, comparing them in relation to efficiency, exactness and applicably. None of these methods is both efficient and exact.

Table 1: Comparison of Full, Sampling, and Iterative Methods

Method	Efficiency	Exactness	Usage
Full	poor	perfect	generalised
Sampling	excellent	approximate	generalised
Iterative	excellent	approximate	interval

1.3 Reversal of Blurring

Mendel and John ([4], page 118) describe the transformation from a type-1 to a type-2 membership function by a process of blurring:

> Imagine blurring the type-1 membership function ... by shifting the points ... either to the left or the right, and not necessarily by the same amounts, ... Then, at a specific value of x, say x', there no longer is a single value for the membership function (u'); instead the membership function takes on values wherever the vertical line $[x = x']$ intersects the blur. These values need not all be weighted the same; hence, we can assign an amplitude distribution to all of these points. Doing this for all $x \in X$, we create a three-dimensional membership function — a type-2 membership function — that characterizes a type-2 fuzzy set.

The collapsing method of defuzzification is a response to the challenge of reversing blurring. Further motivation for this research was to devise an efficient *and* exact type-2 defuzzification technique.

1.4 Embedded Sets

An interesting feature of the FIS is that embedded sets only appear during the final defuzzification stage. (Theoretically they could be employed in the earlier stages, but to do so would be impractical.) A defuzzification technique that reversed blurring would obviate the need for processing embedded sets, which would be an enormous practical advantage, as well as being satisfying from a theoretical perspective.

Though the collapsing algorithm does not require processing of embedded sets, it involves the creation, and subsequent defuzzification, of a single type-1 set which is embedded. Moreover, the embedded set concept is used in the proof of the theorem (section 1) upon which the method is based.

1.5 Overview

Our research sees developing *generalised* type-2 systems as a challenge for the research community. However in this community the interval case is often explored before the generalised case. This paper reports the first results on the interval case with a view to future extension of the research to the generalised case. We propose a straightforward, exact, iterative defuzzification technique for a discretised interval type-2 fuzzy set.

1.6 Preliminaries

The discussion in this article relies on certain assumptions and definitions.

Discretisation Technique
It is assumed that the domain is discretised into an arbitrary number m of vertical slices, and the co-domain into 2 slices, which are the end-points of the secondary domain.

Defuzzification Method
The analysis presented in this paper presupposes that the centroid method of defuzzification is adopted.

Scalar Cardinality
The concept of *scalar cardinality* is frequently encountered in the following analysis. For type-1 fuzzy sets, Klir and Folger ([3], p17) define scalar cardinality as follows:

> The *scalar cardinality* of a fuzzy set A defined on a finite universal set X is the summation of the membership grades of all the elements of X in A. Thus,
>
> $$\mid A \mid = \sum_{x \in X} \mu_A(x).$$

The symbol '$\sum$' as used here represents 'sum'.

To distinguish scalar cardinality from cardinality in the classical sense of the number of members of a set, we adopt the '$\| \ \|$' symbol for scalar cardinality, i.e. $\|A\|$ represents the scalar cardinality of A.

2 The Representative Embedded Set

In this section we introduce the idea of the representative embedded set, a concept which fundamentally underpins the collapsing method of defuzzification that is the subject of this paper.

An interval type-2 fuzzy set is a type-2 fuzzy set in which every secondary membership grade takes the value 1. Such a set is completely specified by its footprint of uncertainty ([4], definition 5, page 119), since all its secondary grades are by definition equal to 1. In the analysis which follows, to speak in terms of the footprint of uncertainty (FOU) of an interval type-2 fuzzy set is equivalent to referring to the interval set itself.

2.1 The Collapsing Technique

It is helpful to think of the interval FOU as a *blurred* type-1 membership function ([4], page 118). The *collapsing technique* is the reversal of this process to create a type-1 fuzzy set from an interval FOU. The type-1 set's membership function is derived so that its defuzzified value is equal to that of the interval set. It is a simple matter to defuzzify a type-1 set, and to do so would be to find the defuzzified value of the original interval set. Hence the collapsing process reduces the computational complexity of interval type-2 defuzzification. We term this special type-1 set the *representative embedded set*. It is a *representative set* because it has the same defuzzified value as the original interval type-2 fuzzy set. It is an embedded set because it lies within the FOU of the interval type-2 fuzzy set. Before defining the representative embedded set, we first define the representative set.

Definition 1 (Representative Set)**.** *A type-1 fuzzy set is a* representative set (RS*) of an interval type-2 fuzzy set if it has the same defuzzified value as the interval set.*

Definition 2 (Representative Embedded Set)**.** *Let $\tilde{F}$ be an interval type-2 fuzzy set with defuzzified value $X_{\tilde{F}}$. Then type-1 fuzzy set R is a* representative embedded set (RES) *of $\tilde{F}$ if its defuzzified value (X_R) is equal to that of $\tilde{F}$, i.e. $X_R = X_{\tilde{F}}$,* and *its membership function lies within the FOU of $\tilde{F}$.*

Clearly an RS may be embedded, i.e. an RES, or non-embedded.

2.2 Derivation of an RES

We know that any RES will lie within the FOU of the interval type-2 fuzzy set it represents. The objective of this analysis is to derive an expression for the membership function of an RES in terms of the upper and lower membership functions of the interval set to be defuzzified. Our strategy is two-stage:

1. We derive a formula for the special case of the interval FOU which has only one blurred vertical slice.

2. We generalise this formula to the typical interval FOU in which every point of the original type-1 membership function has been blurred, i.e. one for which the upper membership function is greater than the lower membership function for every domain value.

3 Solitary Collapsed Slice Lemma (SCSL)

In this section we concentrate on the derivation of an RES in the special case of an interval FOU formed by (upwardly) blurring the membership function of a type-1 fuzzy set at a single domain value x_I (figure 1), to create a vertical slice which is an interval as opposed to a point. This interval is the secondary domain at x_I. The FOU formed by this blurring is depicted in figure 2, and consists of the shaded triangular region plus the line L. We derive a formula for the RES of this somewhat unusual interval FOU in terms of the original type-1 membership function and the amount of blurring.

Type-1 Set with a Single Blurred Grade
Let A be a type-1 fuzzy set that has been discretised into m vertical slices. We calculate the defuzzified value of A, X_A, by finding the centroid of A:

$$X_A = \frac{\sum_{i=1}^{i=m} \mu_A(x_i)x_i}{\sum_{i=1}^{i=m} \mu_A(x_i)} = \frac{\sum_{i=1}^{i=m} \mu_A(x_i)x_i}{\|A\|}.$$

To avoid division by 0, it is assumed throughout this paper that $\|A\| > 0$.

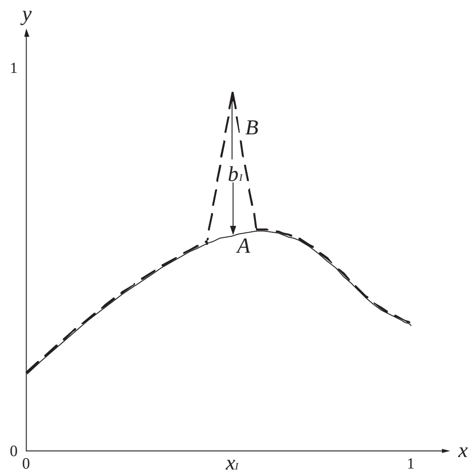

Figure 1: At $x = x_I$ the membership function of type-1 fuzzy set A has been blurred, increasing the membership grade by the amount b_I, creating a new type-1 fuzzy set B.

Now suppose the membership function of A is blurred upwards at domain value x_I, so that x_I, instead of corresponding to the point $\mu_A(x_I)$, corresponds to the co-domain range $[\mu_A(x_I), \mu_A(x_I) + b_I]$. Let B (figure 1) be a type-1 fuzzy set whose membership function is the same as that of A apart from

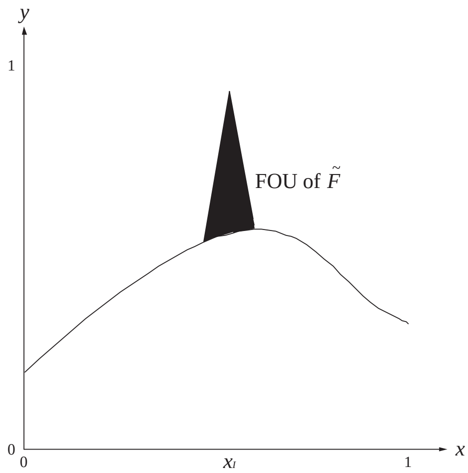

Figure 2: FOU of interval type-2 fuzzy set $\tilde{F}$, which consists of the original line of type-1 set A plus the triangular region.

at the point x_I, for which $\mu_B(x_I) = \mu_A(x_I) + b_I$. X_B, the defuzzified value of B, may be calculated:

$$\begin{aligned} X_B &= \frac{\sum \mu_B(x_i) x_i}{\sum \mu_B(x_i)} \\ &= \frac{\sum \mu_A(x_i) x_i + b_I x_I}{\sum \mu_A(x_i) + b_I} \\ &= \frac{\|A\| X_A + b_I x_I}{\|A\| + b_I} \\ &= X_A + \frac{\|A\| X_A + b_I x_I}{\|A\| + b_I} - X_A \\ &= X_A + \frac{b_I (x_I - X_A)}{\|A\| + b_I}. \end{aligned}$$

Interval Set with One Blurred Slice

Let $\tilde{F}$ (figure 2) be an interval type-2 fuzzy set whose lower membership function is A and upper membership function B. The membership function of A is identical to that of B apart from the blurring at the point x_I which makes $\mu_B(x_I)$ greater than $\mu_A(x_I)$ by the amount b_I.

Full Defuzzification of Set $\tilde{F}$

Full defuzzification requires that *all* the embedded sets of a type-2 set be processed to form the type-reduced set. $\tilde{F}$ contains only two embedded sets, namely A and B. Therefore the type-reduced set of $\tilde{F}$ consists of the two pairs of co-ordinates $(X_A, 1)$ and $(X_B, 1)$. We find the defuzzified value of $\tilde{F}$ by calculating the mean of X_A and X_B, i.e. $\frac{1}{2}(X_A + X_B)$. Let $X_{\tilde{F}}$ be the defuzzified value of $\tilde{F}$. $X_{\tilde{F}}$ will be expressed in terms of $\|A\|$, X_A, x_I and b_I, all of which are known values.

$$\begin{aligned} X_{\tilde{F}} &= \frac{1}{2}(X_A + X_B) \\ &= \frac{1}{2}\left(X_A + X_A + \frac{b_I (x_I - X_A)}{\|A\| + b_I} \right) \\ &= X_A + \frac{b_I (x_I - X_A)}{2(\|A\| + b_I)}. \end{aligned}$$

Type-1 Defuzzification of Set R

Let R be an RES of $\tilde{F}$ such that the membership function of R is the same as that of A for all domain values x_i apart from x_I. At this point the membership function deviates from that of A so that $\mu_R(x_I)$ takes the value $\mu_A(x_I) + r_I$. Figure 3 depicts the membership function of R. Following the same chain of reasoning as in the derivation of X_B, we work out an expression for X_R in terms of $\|A\|$, X_A, x_I and r_I, where $\|A\|$, X_A, x_I are known values:

$$X_R = X_A + \frac{r_I (x_I - X_A)}{\|A\| + r_I}.$$

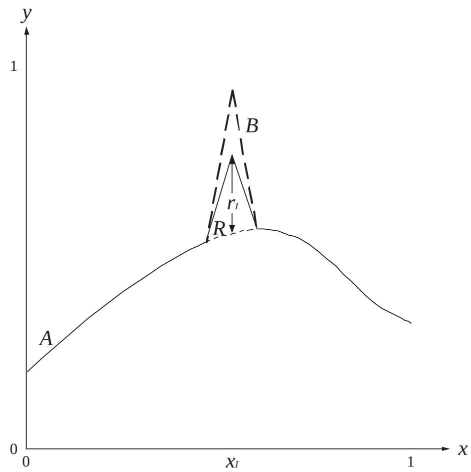

Figure 3: R, the representative embedded set of $\tilde{F}$, is indicated by the undashed line.

Equating X_R and $X_{\tilde{F}}$ to Derive an Expression for r_I

The defuzzified values X_R and $X_{\tilde{F}}$ are by definition equal (section 2), and by equating these values we are able to obtain a formula for r_I in terms of $\|A\|$ and b_I:

$$X_R = X_{\tilde{F}} \Rightarrow r_I = \frac{b_I \|A\|}{2\|A\| + b_I}.$$

We have arrived at the membership function of R, and in so doing proved the solitary collapsed slice lemma:

Lemma 1 (Solitary Collapsed Slice Lemma). *Let A be a discretised type-1 fuzzy set which has been blurred by amount b_I at a single point x_I to form the FOU of interval type-2 fuzzy set $\tilde{F}$. Then R, the RES of $\tilde{F}$, has a membership function such that*

$$\mu_R(x_i) = \begin{cases} \mu_A(x_i) + \dfrac{\|A\| b_I}{2\|A\| + b_I} & \text{if } i = I, \\ \mu_A(x_i) & \text{otherwise.} \end{cases}$$

4 Representative Embedded Set Theorem (REST)

We generalise the solitary collapsed slice lemma to the typical situation in which every point of the type-1 membership function has been blurred. First we present the concept behind the theorem: How an interval type-2 set may be collapsed to create a representative embedded set. We then state the theorem, and go on to prove it inductively.

Collapsing an Interval Set to Form an RES

We have considered an extremely atypical interval FOU whose membership function follows the course of a type-1 fuzzy set apart from at one slice x_I, at which its membership grade opens up into a secondary co-domain $[\mu(x_I), \mu(x_I + b_I)]$. We have done this to provide a simple yet illustrative example of the collapsing process, as a basis for generalisation to the typical interval FOU. The SCSL (section 3) tells us how to calculate the RES for this special case of an interval type-2 set.

Now we proceed to look at the typical interval FOU, in which the upper membership grade is greater than the lower membership grade at every point. The difference between the lower and upper membership grades at any given point is the amount of blur (b_i) at that point, i.e. $\mu_U(x_i) - \mu_L(x_i) = b_i$. The solitary collapsed slice lemma does not apply in this situation. However, this lemma may be applied repeatedly to FOUs assembled in stages using slices taken from the interval set.

Collapsing the 1^{st} FOU to Form RES R_1

The first interval FOU to be collapsed (figure 4) comprises the slice x_1 (at $x = 0$), plus the rest of the lower membership function L, (represented by the shaded triangular region plus the line L.) The lower membership function of the FOU is the line L, and the upper membership function starts (at $x = 0$) at the line U, but immediately descends to L (slice x_2), after which it follows the course of L (slices $x_2 \ldots x_m$). The SCSL tells us that this interval set may be collapsed into its RES R_1, depicted in figure 5. The collapse increases the membership grade $\mu_L(x_1)$ by r_1 to $\mu_{R_1}(x_1)$.

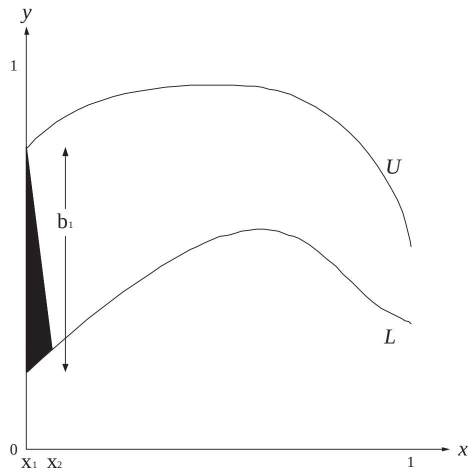

Figure 4: The first slice in interval type-2 fuzzy set $\tilde{F}$.

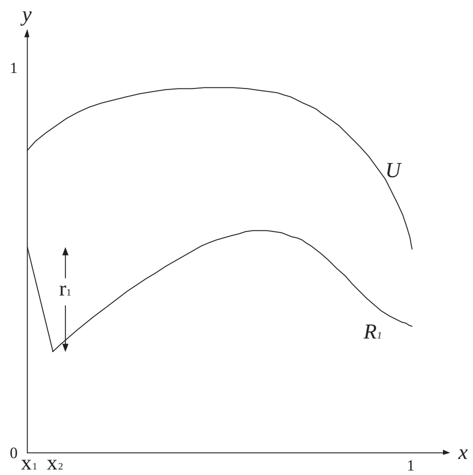

Figure 5: The first slice collapsed, creating an RES R_1 for the interval type-2 fuzzy set $\tilde{F}$.

Collapsing the 2^{nd} FOU to Form RES R_2

We now move on to the second FOU. Figure 6 shows this FOU before it is collapsed. The SCSL is re-applied, but instead of the lower membership function being L, it is now R_1. The RES of the second FOU is R_2, which is depicted in figure 7.

Collapsing the $(k+1)^{th}$ FOU to Form RES R_{k+1}

Suppose FOUs $1, \ldots, k$ have been collapsed in turn, with R_k being the most recently formed RES. Then it is the turn of the $k + 1^{th}$ FOU to be collapsed. The lower membership function is R_k. This situation prior to the $k + 1^{th}$ FOU's collapse is represented in figure 8; the situation after the collapse in figure 9.

Collapsing the m^{th} FOU to Form RES R

Suppose FOUs $1, \ldots, m-1$ have been collapsed

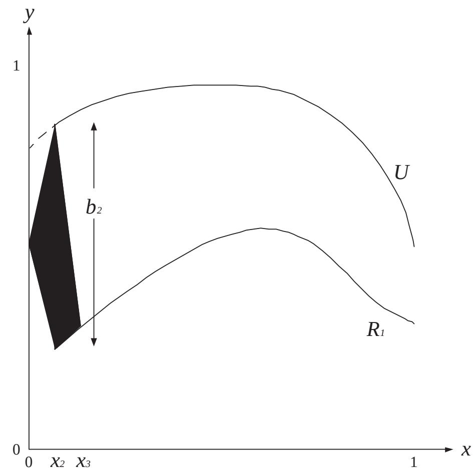

Figure 6: For the interval type-2 fuzzy set $\tilde{F}$, the first slice is collapsed, and the second slice is shown.

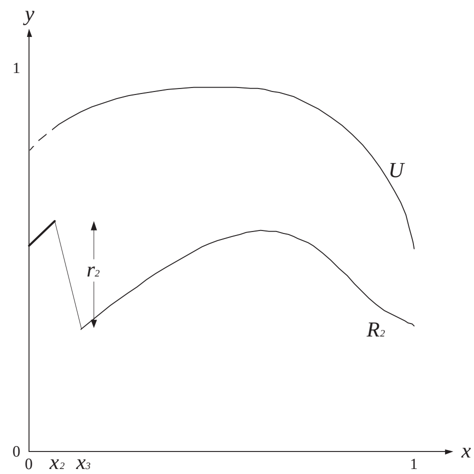

Figure 7: Slices 1 and 2 collapsed, creating RES R_2 for the interval type-2 fuzzy set $\tilde{F}$.

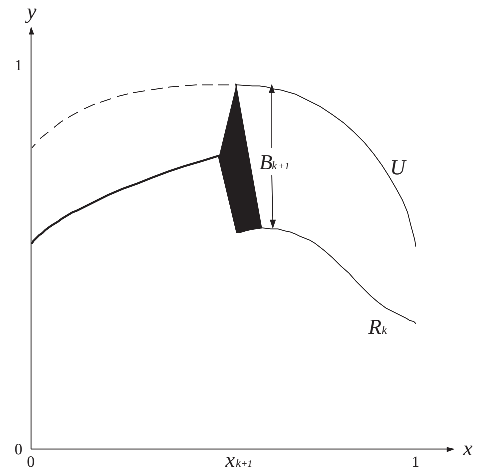

Figure 8: Slices 1 to k collapsed, slice $k+1$ about to be collapsed, for interval type-2 fuzzy set $\tilde{F}$.

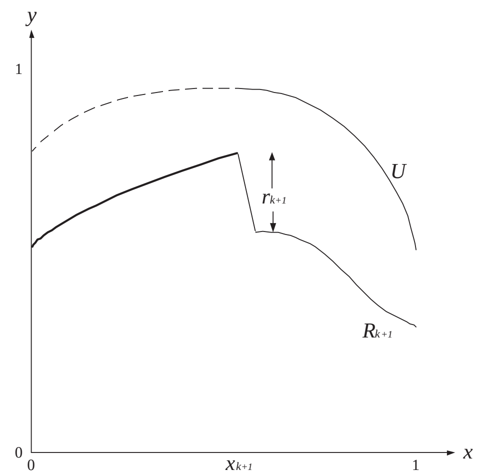

Figure 9: Slices 1 to $k+1$ collapsed, creating RES R_{k+1} for interval type-2 fuzzy set $\tilde{F}$.

in turn, with R_{m-1} being the most recently formed RES. Then it is the turn of the m^{th} FOU to be collapsed. The lower membership function is R_{m-1}, and the slice to be collapsed is slice m at $x = 1$. After the collapse the new lower membership function is R_m. As the m^{th} slice is the final slice, then R_m is the RES of the entire interval type-2 fuzzy set, and is equivalent to *the* RES R.

We now state and prove the representative embedded set theorem.

Theorem 1 (Representative Embedded Set Theorem)**.** *A discretised type-2 interval fuzzy set $\tilde{F}$, having defuzzified value $X_{\tilde{F}}$, lower membership function L, and upper membership function U, may be collapsed into a type-1 fuzzy set R, known as the* representative embedded set*, whose defuzzified value X_R is equal to $X_{\tilde{F}}$, with membership function such that:*

$$\mu_R(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i},$$

where $b_i = \mu_U(x_i) - \mu_L(x_i)$.

Proof. Let $X_{\tilde{F}}$ be an interval type-2 fuzzy set, and $R_1, \ldots R_m$ be type-1 fuzzy sets formed by collapsing vertical slices $1, \ldots m$ inclusive of $X_{\tilde{F}}$, i.e. R_1 is formed by collapsing slice 1, R_2 by collapsing slices 1 and 2, and R_i by collapsing slices 1 to i. R_m is a representative set of $X_{\tilde{F}}$, equivalent to the R of the previous discussion.

Proof by induction on the number of collapsing vertical slices will be used:

Basis (Collapsing the 1^{st} slice to form R_1): Figures 4 and 5 depict the collapse of the first slice. The resultant RES is R_1. For R_1, $i = 1$, and $\sum_1^{i-1} r_j = 0$. We need to prove that

$$\mu_{R_1}(x_1) = \mu_L(x_1) + \frac{\|L\| b_1}{2\|L\| + b_1},$$

but this is actually what we have when we apply the solitary collapsed slice lemma (section 3) for $i = 1$.

Induction hypothesis: Assume the theorem is true for R_k, i.e. that slices $1, \ldots, k$ have been collapsed to form type-1 fuzzy set R_k, and that

$$\mu_{R_k}(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}.$$

(In this formula, for $i > k$, $b_i = 0$.)

Induction Step: Now we collapse slice $(k+1)$, which is a single slice. Applying the SCSL (section 3) to R_k we have:

$$r_{k+1} = \frac{\|R_k\| b_{k+1}}{2\|R_k\| + b_{k+1}}.$$

We need to prove that for all i

$$\mu_{R_{k+1}}(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}.$$

The proof will be split up into three cases.

Case 1: $1 \leq i \leq k$

$$\begin{aligned} \mu_{R_{k+1}}(x_i) &= \mu_{R_k}(x_i) \\ &= \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}. \end{aligned}$$

Case 2: $i = k+1$

$$\begin{aligned} \mu_{R_{k+1}}(x_i) &= \mu_L(x_i) + r_{k+1} \\ &= \mu_L(x_i) + \frac{\|R_k\| b_i}{2\|R_k\| + b_i}. \end{aligned}$$

We know that

$$\begin{aligned} \|R_k\| &= \sum_{j=1}^{m} \mu_{R_k}(x_j) \\ &= \sum_{j=1}^{k} (\mu_L(x_j) + r_j) + \sum_{j=k+1}^{m} \mu_L(x_j) \\ &= \|L\| + \sum_{j=1}^{k} r_j, \end{aligned}$$

and therefore we conclude that

$$\mu_{R_{k+1}}(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{k} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{k} r_j\right) + b_i}.$$

Case 3: $i > k+1$

$$\begin{aligned} \mu_{R_{k+1}}(x_i) &= \mu_{R_k}(x_i) \\ &= \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}. \end{aligned}$$

Conclusion: Drawing the three cases together, we conclude that for all i,

$$\mu_{R_{k+1}}(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}.$$

$\square$

5 Collapsing as Type-Reduction

Conventional type-reduction ([5], pages 248-252), whether of interval or generalised type-2 fuzzy sets, creates the type-reduced set, which is a type-1 set whose domain values are the centroids of all the embedded sets of the original type-2 fuzzy set. The significant point, however, is that a type-1 fuzzy set is produced from a type-2 fuzzy set, which is what happens when an interval set is collapsed.

Both methods of type-reduction result in a type-1 fuzzy set, which is easily defuzzified. However conventional type-reduction is in itself so computationally complex that the fact that the process creates an easily defuzzified type-1 set is spurious. Moreover, it is far less computationally complex to defuzzify an RES than a TRS, as an RES has far fewer points to process than its equivalent TRS.

6 Pseudo Representative Embedded Set

The SCSL (section 3) tells us that:

$$r_I = \frac{\|A\| b_I}{2\|A\| + b_I}.$$

We have assumed (section 3) that $\|A\| > 0$. Therefore, by dividing the numerator and denominator by $\|A\|$, we can deduce that

$$r_I = \frac{b_I}{2 + \frac{b_I}{\|A\|}} < \frac{b_I}{2}.$$

As $\|A\|$ increases, r_I increases and approaches $\frac{b_I}{2}$. $\frac{b_I}{2}$ can therefore be considered an approximation for r_I. This approximation makes sense intuitively.

The REST (section 1) states that:

$$\mu_R(x_i) = \mu_L(x_i) + \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}.$$

The expression $\|L\| + \sum_{j=1}^{j=i-1} r_j$ replaces $\|A\|$ in the SCSL. The same line of reasoning that was applied to the blurred single point case applies equally here. As each slice is collapsed, $\|L\| + \sum_{j=1}^{j=i-1} r_j$ increases, which means that as the collapse progresses, the r_i for each collapsed slice is a closer approximation to $\frac{1}{2}b_i$.

Definition 3 (Pseudo Representative Embedded Set)**.** *For an interval type-2 fuzzy set, the* pseudo representative embedded set (PRES) *is the type-1 fuzzy set whose membership function at every point takes the mean value of the lower membership function and the upper membership function of the interval set. Symbolically* $\mu_P(x_i) = \frac{1}{2}\left(\mu_L(x_i) + \mu_U(x_i)\right)$.

The approximation of the PRES to a RES is equivalent to the approximation involved in equating r_i to $\frac{1}{2}b_i$, given that for all vertical slices, $r_i < \frac{1}{2}b_i$.

7 Conclusions and Further Work

The collapsing method of defuzzification generates a type-1 embedded set from an interval type-2 fuzzy set that is representative in that it has the same defuzzified value as the original type-2 set. The representative embedded set may be thought of as a type-reduced set, but the collapsing form of type-reduction is vastly more efficient than the conventional, full type-reduction. The collapsing method promises to be efficient and accurate in its implementation.

Future work will consider the following:

Testing the Collapsing Method This method requires testing, to quantify both the time saving it makes possible and confirm its accuracy. Tests may be performed for either defuzzification in isolation, or defuzzification as part of an interval type-2 FIS.

Discretisation with More Than 2 Co-Domain Slices The research presented in this paper concerns the situation where the co-domain is sliced at 2 points for every domain value, at the lower and upper membership grades for that domain value. It would be desirable to extend the collapsing method to cover discretisation in which the co-domain is sliced more than twice.

Other Methods of Defuzzification There is no obvious reason why this technique may not be extended to other methods of defuzzification besides the centroid method.

Generalised Type-2 Fuzzy Sets The technique as described in this paper applies to interval type-2 fuzzy sets; we plan to extend it to generalised sets.

The PRES as a Substitute for an RES It is far easier to calculate the PRES membership function than that of an RES, but of course the defuzzified value of the PRES is only an approximation. It would be useful to investigate (both mathematically and experimentally) the extent to which accuracy is lost in replacing an RES by the PRES.

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